arXiv:math/0312080v1 [math.DG] 3 Dec 2003

The Plateau problem at infinity for horizontal ends and genus 1

Laurent Mazet

Abstract

In this paper, we study Alexandrov-embedded r-noids with genus 1 and horizontal ends. Such minimal surfaces are of two types and we build several examples of the first one. We prove that if a polygon bounds an immersed polygonal disk, it is the flux polygon of an r-noid with genus 1 of the first type. We also study the case of polygons which are invariant under a rotation. The construction of these surfaces is based on the resolution of the Dirichlet problem for the minimal surface equation on an unbounded domain.

2000 *Mathematics Subject Classification.* 53A10.
Keywords: Minimal Surface, Dirichlet Problem, Boundary Behaviour, Degree theory.

Introduction

The classical Plateau problem consists in finding a surface of least area bounded by a given closed curve in $\mathbb{R}^3$, such a surface satisfies that its mean curvature vanishes. A surface in $\mathbb{R}^3$ with zero mean curvature will be called a minimal surface. The Plateau problem has been solved by T. Radó in 1930. A generalization of this problem is: finding a minimal surface for a given asymptotic behaviour. We first give sense to this question.

We know that, if a complete minimal surface M has finite total curvature and r embedded ends (such a surface is called an r-noid), each end of this minimal surface is asymptotic to a plane or to a half-catenoid; besides, we can associate to each end a vector in $\mathbb{R}^3$, this vector is called the flux vector of the end. These vectors satisfy the following condition: the sum of the flux vectors over all ends is zero. So the generalization of the Plateau problem is: given a finite number of vectors such that their sum is zero, can we find an r-noid which has these vectors as flux vectors? This problem is called

the Plateau problem at infinity. Besides, we know that M is conformally equivalent to a compact Riemann surface $\overline{M}$ minus r points, the punctures of M, and what we call the genus of M is in fact the genus of $\overline{M}$.

In this paper, we shall study a particular case of this problem. In [CR], C. Cosín and A. Ros give a description of the space of solutions of the Plateau problem at infinity with an asymptotic behaviour which is symmetric with respect to an horizontal plane (*i.e.* all the flux vectors are horizontal) for genus 0 (the Riemann surface $\overline{M}$ is the Riemann sphere $\mathbb{S}^2$). In the genus 0 case, there is a natural order on the ends. Then, since the flux vectors are horizontal and their sum is zero, the flux vectors draw a polygon in $\mathbb{R}^2$; this polygon is called the flux polygon of M. C. Cosín and A. Ros give a necessary and sufficient condition on this polygon for having a solution to the Plateau problem at infinity. In our work, we study the case where $\overline{M}$ is of genus 1 (*i.e.* $\overline{M}$ is a torus).

For the case of genus 1, we need to distinguish two types of r-noid with horizontal ends; this classification depends on the place of the punctures on the torus: when M is of the first type, there is a natural order on the punctures and for the second type, there is not. If M is an r-noid of genus 1 and horizontal ends of the first type, since there is a natural order on the punctures, we can define as in the genus 0 case the flux polygon associated to M. Then our main result can be state as follow (see Theorem 2).

Let M be an r-noid with genus 0 and horizontal ends, then there exists Σ an r-noid with genus 1 and horizontal ends of the first type which have the same flux polygon as M.

In Corollary 4, we give examples of polygons which are the flux polygons of r-noid with genus 1 but not the flux polygons of r-noid with genus 0.

The r-noids M, we consider, are symmetric with respect to a horizontal plane that we can normalized to be $\{z = 0\}$. Then to build them, it is sufficient to build the part M^+ of M included in $\mathbb{R}^2 \times \mathbb{R}_+$. The proof of our result is then based on the fact that M^+ is the conjugate of a minimal surface that can be seen as the graph of a function u over a planar "domain" which depends on the flux polygon of M. This "domain" is in fact a multi-domain (see definitions in Section 1).

If Ω is a domain in $\mathbb{R}^2$ and u is a function on Ω, the graph of u is a minimal surface in $\mathbb{R}^3$ iff u satisfies the elliptic partial equation called the

minimal surface equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0 \tag{MSE}$$

Like every partial differential equation, we can associate to (MSE) the Dirichlet problem that consists in finding a function u on Ω which is a solution of the minimal surface equation and takes on assignated values on the boundary of Ω.

The first step to build a r-noid with genus 1 having a polygon V as flux polygon is then to consider a multi-domain Ω associated to this polygon and solve a Dirichlet problem on Ω. The boundary data is such that the graph of the solution can be the conjugate of M^+. As an example if V is a triangle ABC, we can glue along the three edges of V three half-strips ($[A,B]\times\mathbb{R}_+$ along $[A,B]$, $[B,C]\times\mathbb{R}_+$ along $[B,C]$...), we get an unbounded domain D. Then the multi-domain Ω is the universal cover of $D\backslash\{Q\}$ where Q is a point in the triangle ABC. The boundary value we take on Ω is $\pm\infty$ alternating the sign such that for every half-strip in Ω one side has $+\infty$ and the other has $-\infty$.

If Ω is a multi-domain associated to the polygon V, the conjugate surface to the graph of the solution of the Dirichlet problem is a minimal surface which is invariant under a translation by a horizontal vector. If this vector is non zero, this minimal surface is not the piece M^+ of an r-noid M, then we must choose a multi-domain associated to V such that the corresponding vector is zero. In our example of the triangle, the only choice we have is the position of the point Q in ABC, then we need to prove that there exists $Q\in ABC$ such that the corresponding vector vanishes. This problem is called the period problem.

In the case of the triangle, the idea to solve the period problem is the following. To each point Q in the interior of the triangle ABC, we can associate an horizontal vector, in fact, this defines a continuous map from the triangle to $\mathbb{R}^2$ and we want to show that this map vanishes; this map is the period map. Then we whall compute the degree of the period map along the boundary of the triangle and find a non-zero number, then the period map must vanishes at one point. The proof of our main result is based on a generalization of this arguement.

The paper is organized as follows; in the first section, we define all the notions of multi-domain we use in the following and the objects associated to a function on such domains. We also precise what kind of r-noid we study

in this paper and give the first results concerning them.

Section 2 is devoted to the resolution of the Dirichlet problem on a multi-domain Ω associated to a polygon V. In this section, we make use of tools developped in [Ma1] and recalled in Appendix A.

In the third section, we study the regularity of the graph of the solution of the Dirichlet problem of the preceding section near the singularity point of the multi-domain Ω.

In Section 4, we explain what is the period problem for the construction of a r-noid. We also generalize the notion of the period map and give the proof of our main result (Theorem 2) in using Theorem 7 which is proved in Section 5.

Section 5 is devoted to an extension of the period map and the computation of its degree.

In section 6, we give examples of r-noids with genus 1 that are not given by Theorem 2. In particular, we consider the case where the polygon V is a regular polygon.

Let us fix some notations. In the following, when u is a function on a domain of $\mathbb{R}^2$ we shall note $W = \sqrt{1 + |\nabla u|^2}$. We shall also use the classical following notations for partial derivatives: $p = \dfrac{\partial u}{\partial x}$ and $q = \dfrac{\partial u}{\partial y}$. Besides, for the graph of u, we shall always chose the downward pointing normal to give an orientation to the graph.

1 Preliminaries

1.1 Graph on multi-domains

In this section,we give several generalizations of the notion of domain in $\mathbb{R}^2$, since our aim is to describe some minimal surfaces as the graph of a function which is a solution of (MSE) (see the example of the half helicoid). We use notions introduced in [CR] and [Ma1].

Let us consider a pair (D, ψ) where D is 2-dimensionnal flat manifold with piecewise smooth boundary and $\psi : D \to \mathbb{R}^2$ is a local isometry. The map ψ is called the *developing map* and the points where the boundary ∂D is not smooth are called vertices. If a part of the boundary of D is linear then this part will be called an *edge* of the boundary.

Definition 1. *A pair (D, ψ), where D is a simply-connected 2-dimensionnal complete flat manifold with piecewise smooth boundary and $\psi : D \to \mathbb{R}^2$ is a local isometry, is a* multi-domain *if each connected component of the smooth part of ∂D is a convex arc.*

Let D be a complete metric space and Q a point of D we say that D admits a *cone singularity* at Q of angle α if $D\backslash\{Q\}$ is a 2-dimensional flat manifold and if there exist $\rho_0 > 0$ such that $\{M \in D|\ d(M,Q) < \rho_0\}$ is isometric to $\{(\rho,\theta)|\, 0 \leq \rho < \rho_0,\ 0 \leq \theta \leq \alpha\}$ with the polar metric $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2$ where all the points $(0,\theta)$ are identified and where for every ρ we identified $(\rho,0)$ with (ρ,α) (the isometry sends Q to $(0,0)$).

Definition 2. *A triplet* (D,Q,ψ) *is a* multi-domain with a cone singularity at Q *if*

1. *D is a simply-connected complete metric space,*
2. *there exists $q \in \mathbb{N}$ such that D admits a cone singularity at Q of angle $2q\pi$,*
3. *D has piecewise smooth convex boundary and*
4. *$\psi : D \to \mathbb{R}^2$ is a local isometry outside Q.*

We can remark that a multi-domain (D,ψ) can be seen as a multi-domain with a cone singularity at Q if Q is some point of D. The angle at the singularity is 2π.

Let (D,ψ) be a compact multi-domain such that its boundary is only composed of edges. The developing map allows us to see ∂D included in $\mathbb{R}^2$ since there are only edges $\psi(\partial D)$ is a polygon in $\mathbb{R}^2$. The same thing can be done for (D,Q,ψ) a multi-domain with cone singularity. We then say that a polygon V bounds a multi-domain (with perhaps a cone singularity) if there exists (D,ψ) or (D,Q,ψ) such that $V = \psi(\partial D)$. When V bounds a multi-domain (D,ψ) we shall also say that v bounds an immersed polygonal disk as in [CR].

The last generalization we need is to give a sense to a cone singularity with infinite angle.

Let us consider $\mathcal{D} = \{(\rho,\theta)|\ \rho \in \mathbb{R}_+,\ \theta \in \mathbb{R}\}$ with the polar metric and where all the points $(0,\theta)$ are identified, this point will be called the vertex of $\mathcal{D}$ and noted $\mathcal{O}$. The space $\mathcal{D}$ is a simply-connected complete metric space and $\mathcal{D}\backslash\mathcal{O}$ is a 2-dimensional flat manifold.

Definition 3. *A triplet* $(\Omega,\mathcal{Q},\varphi)$ *is a* multi-domain with a logarithmic singularity at $\mathcal{Q}$ *if*

1. *Ω is a simply-connected complete metric space,*
2. *$\mathcal{Q} \in \Omega$,*

3. *$\Omega\backslash\mathcal{Q}$ is a 2-dimensional flat manifold with piecewise smooth convex boundary,*

4. *$\varphi : \Omega \to \mathcal{D}$ is a local isometry such that $\varphi(\mathcal{Q}) = \mathcal{O}$ and*

5. *there exist a neighborhood $\mathcal{N}$ of $\mathcal{Q}$ in Ω and $\rho > 0$ such that $\varphi|_{\mathcal{N}}$ is an isometry into $\{M \in \mathcal{D} \mid d(\mathcal{O}, M) < \rho\}$.*

Let us define $R_\alpha : \mathcal{D} \to \mathcal{D}$ by $R_\alpha(r, \theta) = (r, \theta + \alpha)$, R_α is an isometry of $\mathcal{D}$.

Definition 4. *A multi-domain with a logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$ is* periodic *if there exists $f : \Omega \to \Omega$ an isometry and $n \in \mathbb{N}^*$ such that*

$$\varphi \circ f = R_{2n\pi} \circ \varphi \qquad (*)$$

The period of Ω is then $2\pi q$ where q is the smallest n such that there exists f making $(*)$ true.

The first example of a multi-domain with a logarithmic singularity, we can give, is $(\mathcal{D}, \mathcal{O}, \mathrm{id})$. This multi-domain is periodic of period 2π.

Construction 1. Let us consider (D, ψ) a multidomain and Q a point in D. We then note $\Omega \xrightarrow{\pi} D\backslash Q$ a universal cover of $D\backslash Q$. We can pull back to Ω the flat metric of D. The metric completion of Ω is just $\Omega \cup \{\mathcal{Q}\} = \overline{\Omega}$ where $\mathcal{Q}$ is a point "above" Q (*ie* if $\mathcal{A}_n \to \mathcal{Q}$, we have $\pi(\mathcal{A}_n) \to Q$). If (ρ, θ) are the polar coordinates on $\mathbb{R}^2$ of center $\psi(Q)$ then, on Ω the 1-forms $(\pi \circ \psi)^* \mathrm{d}\rho$ and $(\pi \circ \psi)^* \mathrm{d}\theta$ are exact and by integration we can define a map $\varphi : \Omega \cup \{\mathcal{Q}\} \to \mathcal{D}$ such that $(\overline{\Omega}, \mathcal{Q}, \varphi)$ is a multi-domain with a logarithmic singularity. The multi-domain, we have just build, is a periodic one of period 2π.

In fact, we can do the same work for (D, Q, ψ) a multi-domain with a cone singularity at Q of angle $2q\pi$. We get $(\overline{\Omega}, \mathcal{Q}, \varphi)$ a periodic multi-domain with a logarithmic singularity (the period is less than $2q\pi$) and a covering map $\pi : \overline{\Omega} \to D$ with $\pi(\mathcal{Q}) = Q$.

Construction 2. The inverse construction is also possible. Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with a logarithmic singularity of period $2q\pi$ and isometry f. Then by taking the quotient of Ω by the group $\{f^n\}_{n \in \mathbb{Z}}$, we build a multi-domain with a cone singularity at Q, the image of $\mathcal{Q}$ in the quotient, and angle $2q\pi$.

Remark 1. We make a remark about these two constructions. Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with a logarithmic singularity of period $2q\pi$ and isometry f. If we considere the quotient of Ω by the group $\{f^{an}\}_{n \in \mathbb{Z}}$ for

$a \in \mathbb{N}^*$, we get a multi-domain with a cone singularity (D, Q, ψ). The cone singularity at Q of D has $2qa\pi$ as angle. But if we apply Construction 1 to D we get Ω which have a period less than $2qa\pi$ if $a > 1$.

Let $V = (v_1, \ldots, v_r)$ be a polygon which, for example, bounds an immersed polygonal disk (D, ψ). If $Q \in D$, we make Construction 1 and we get a multi-domain with logarithmic singularity $(\overline{\Omega}, \mathcal{Q}, \varphi)$. The quotient (D', Q', φ') of $\overline{\Omega}$ by $\{f^{2n}\}_{n\in\mathbb{Z}}$ is a multi-domain with cone singularity of angle 4π; besides this multi-domain bounds the polygon $(v_1, \ldots, v_r, v_1, \cdots, v_r)$. But since Construction 1 gives $\overline{\Omega}$ for D and D', the two polygons V and $(v_1, \ldots, v_r, v_1, \cdots, v_r)$ will not be distinguished in the following.

Let $(\Omega, \mathcal{Q}, \varphi)$ be a multi-domain with logarithmic singularity and A be a point in $\mathbb{R}^2$. We then can define the map $\varphi_A : \Omega \to \mathbb{R}^2$ by $\varphi_A = G \circ \varphi$ with $G(\rho, \theta) = A + (\rho\cos\theta, \rho\sin\theta)$. If Ω is given by Construction 1 we always choose $A = \psi(Q)$.

Let Ω such that (Ω, φ) or (Ω, Q, φ) corresponds to one of the three definitions of multi-domain given above. Let u be a function defined on Ω or Ω minus its singularity. The *graph* of u is then the surface in $\mathbb{R}^3$ defined by $\{\varphi(x), u(x)\}_{x\in\Omega}$ or $\{\varphi_A(x), u(x)\}_{x\in\Omega\setminus\{\mathcal{Q}\}}$ with $A \in \mathbb{R}^2$. In the following the function u will often be a solution of the minimal surface equation, in this case the graph of u becomes a minimal surface of $\mathbb{R}^3$. The fact that u is a solution of (MSE) allows us to define a closed 1-form $\mathrm{d}\Psi_u$ on Ω, $\mathrm{d}\Psi_u$ is the inner product $\frac{\nabla u}{W} \lrcorner \mathrm{d}V$ where $\mathrm{d}V$ is the volume form on Ω. Since $\mathrm{d}\Psi_u$ is closed we can define locally a function Ψ_u (obviously Ψ_u is well defined only if we fix its value at one point). Ψ_u is locally defined in the interior of Ω minus the possible singularity, but Ψ_u is 1-Lipschitz countinuous then it can be countinuously extended to the singularity and the boundary, then since Ω is simply connected Ψ_u is then globally defined on Ω. In fact Ψ_u correponds to the third coordinates of the conjugate surface to the graph of u; Ψ_u is called the conjugate function to u. For other properties on Ψ_u we refer to [JS] and Appendix A.

In [JS] and [Ma1], we can find the most general answer to the Dirichlet problem on compact multi-domain (D, ψ): the Dirichlet problem consists in finding a solution u on D of (MSE) knowing its value on the boundary.

On multi-domain with cone or logarithmic singularity there is no general answer. To give an exemple of solution of (MSE) on a multi-domain with logarithmic singularity, let us consider the function u defined on $(\mathcal{D}, \mathcal{O}, \mathrm{id})$ by $u(\rho, \theta) = \theta$, it is obvious that the graph of u is the half of an helicoid; more precisely it is the surface given in isothermal coordinate by $(a, b) \mapsto (\sinh a \cos b, \sinh a \sin b, b)$ for $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$. The function u is then a

solution of the minimal surface equation.

1.2 The r-noids of genus 1

In this section we give precise definitions of r-noids, flux at one end and other objects linked to the Plateau problem at infinity. We also give some results concerning this problem and explain how we can build solutions in the genus 1 case.

Let M be a complete minimal surface with finite total curvature in $\mathbb{R}^3$; we know that M is isometric to a compact Riemann surface $\overline{M}$ minus a finite number of points (we can refer to [Os]). Then M has a finite number of annular ends; when these ends are embedded they are asymptotic either to a half-catenoid or to a plane. A properly immersed minimal surface with r embedded ends will be called a *r-noid*. We can associate to each end a vector which caracterizes the direction and the growth of the asymptotic half-catenoid (when the end is asymptotic to a plane this vector is zero); this vector is called the *flux* of the end (for a precise definition of the flux see [HK]). If $v_1, \ldots, v_r$ are the fluxes at each end, we have the following balancing condition:

$$v_1 + \cdots + v_r = 0 \tag{1}$$

This condition tells us that the total flux of the system vanishes. If $v_1, \ldots, v_r$ are vectors in $\mathbb{R}^3$ such that (1) is verified and g is a non-negative integer, the Plateau problem at infinity for these data is to find an r-noid of genus g which has $v_1, \ldots, v_r$ as fluxes at its ends (the genus g is the genus of $\overline{M}$).

Let $X : M \longrightarrow \mathbb{R}^3$ be an r-noid. M is conformally equivalent to a compact surface $\overline{M}$ minus r points $p_1, \ldots, p_r$. We will say that M is *Alexandrov-embedded* if $\overline{M}$ bounds a compact 3-manifold $\overline{\Omega}$ and the immersion X extends to a proper local diffeomorphism $f : \overline{\Omega} \backslash \{p_1, \ldots, p_r\} \longrightarrow \mathbb{R}^3$. An Alexandrov-embedded surface has a canonical orientation; we choose the Gauss map to be the outward pointing normal. An Alexandrov-embedded r-noid can not have a planar end (see [CR]).

We are interested in the case where $X : M \longrightarrow \mathbb{R}^3$ is an Alexandrov-embedded r-noid of genus g and r horizontal ends (*ie* the flux at each end is an horizontal vector).

Let $X : M \longrightarrow \mathbb{R}^3$ be a nonflat immersion of a connected orientable surface M and Π be a plane in $\mathbb{R}^3$, normalized to be $\{x_3 = 0\}$. We note by

S the Euclidiean symmetry with respect to Π and consider the subsets:

$$M^+ = \{p \in M | x_3(p) > 0\}$$
$$M^- = \{p \in M | x_3(p) < 0\}$$
$$M^0 = \{p \in M | x_3(p) = 0\}$$

With these notation we have:

Definition 5. *We shall say that M is strongly symmetric with respect to Π if*

- *There exists an isometric involution $s : M \longrightarrow M$ such that $\psi \circ s = S \circ \psi$.*
- *$\{p \in M | s(p) = p\} = M^0$.*
- *The third coordinate N_3 of the Gauss map of M takes positive (resp. negative) values on M^+ (resp. M^-).*

In [CR], C. Cosín and A. Ros prove

Proposition 1. *Let M be an r-noid with horizontal ends. Then M is strongly symmetric with respect to an horizontal plane if and only if M is Alexandrov-embedded.*

The notion of strong symmetry is then important for the study of the Alexandrov-embedded r-noid.

The case of genus 0 was studied by C. Cosín and A. Ros in [CR]; they show that in this case there is a natural order on the ends. Let M be an Alexandrov-embedded r-noid of genus 0 if $2v_1, \dots, 2v_r$ are the fluxes of M ordered as the ends, the polygon $(v_1, \cdots, v_r)$ is called the *flux polygon* of M and is noted $F(M)$. We then have

Theorem 1. *Let $v_1, \dots, v_r$ be horizontal vectors such that $v_1 + \cdots + v_r = 0$ and V the associated polygon, then there exists M an Alexandrov-embedded r-noid of genus 0 such that $F(M) = V$ if, and only if, V bounds an immersed polygonal disk. Besides there is a bijection from the set of M such that $F(M) = V$ and the set of the immersed polygonal bounded by V.*

In the following, when $(\mathcal{P}, \psi)$ is an immersed polygonal disk we shall call $\Sigma(\mathcal{P})$ the r-noid of genus 0 associated to $\mathcal{P}$ by this bijection; we refer to [CR] and [Ma1] for more explanations on this theorem.

In this paper we are interested in the case of genus 1. Let M be an Alexandrov-embedded r-noif of genus 1 with horizontal ends. M is conformally a torus $\overline{M}$ minus r points $p_1, \dots, p_r$. By Proposition 1, M is strongly symmetric with respect to the plane $\{z = 0\}$ (in fact M is strongly symmetric with respect to an horizontal plane that we can normalize to be $\{z = 0\}$). As in [CR], the punctures p_i are fixed by the antiholomorphic involution s given by Definition 5; we have the following lemma.

Lemma 1. *Let $\overline{M}$ be a conformal torus and s an antiholomorphic involution on $\overline{M}$. We suppose that s has a fixed point then the set of the fixed points of s is two separated circles.*

We can then give the following definition.

Definition 6. *Let M be an Alexandrov-embedded r-noid with genus one and horizontal ends and p_i the punctures of M. We note s the antiholomorphic involution associated to M, we then have C_1 and C_2 the two circles of fixed points of s. With this notations, we shall say that:*

- *M is of* type I *if all the p_i are in one of the two circles C_1 and C_2,*
- *M is of* type II *if not.*

We suppose now that M is of type I, the circles of fixed points are the boundary of $M^+ \cup M^0$, this minimal surface is oriented by the outward pointing normal and then its boundary has a natural orientation. Then the circle that contains the points p_i has a natural orientation and we suppose that the points p_i are numbered with respect to this orientation. We note $2v_i$ the flux vector associated to p_i. We know that we have $\displaystyle\sum_{i=1}^{r} v_i = 0$, so, as in the case $g = 0$, the list $(v_1, \dots, v_r)$ defines a polygon which we call the *flux polygon* of M and we note it $F(M)$. We can remark that if M is of type II we do not have such an easy definition.

In the following, we use some arguements developped by C. Cosín and A. Ros in [CR] to the study of the genus 0 case. Let $X : \overline{M} \backslash \{p_1, \dots, p_r\} \to M$ be an Alexandrov-embedded r-noid with genus one and horizontal ends of type I. The surface $M^+ \cup M^0$ is topologically an annulus, so by passing to the universal cover, we can define its conjugate surface $M^*_{0,+}$. The surface $M^*_{0,+}$ is a periodic minimal surface and its period vector is $\displaystyle\int_\gamma \mathrm{d}X^*$ where γ is a path in $M^+ \cup M^0$ that generates $\pi_1(M^+ \cup M^0)$. Since M is of type I all the p_i are in one circle of fixed points, then the other circle of fixed points

generates $\pi_1(M^+ \cup M^0)$ and since the normal along this circle is horizontal the first two coordinates of the period vector are zero. This proves that the period vector is a vertical vector.

The symmetry curves M^0 consists of r complete strictly convex curves in the plane $\{z = 0\}$ (they are the images by X of the arcs $p_i p_{i+1}$) and the image by X of the circle of fixed points that contains no p_i. In the conjugate surface, these curves are transformed in vertical lines; the image of the arc $p_i p_{i+1}$ in the conjugate surface are vertical straight-lines over the vertices of the polygon $F(M)$

Since on M^+ the third coordinate of the normal is positive, the projection map Π from M_+^*, the conjugate of M^+, into the plane $\{z = 0\}$ is a local diffeomorphism. So we can pull back to M_+^* the flat metric of this plane. Besides M_+^* is stable by the translation t of vector $\displaystyle\int_\gamma \mathrm{d}X^*$. Since this vector is vertical, $\Pi \circ t = \Pi$; t is then an isometry of M_+^* with the flat metric $\Pi^*(\mathrm{d}s^2_{\mathbb{R}^2})$.

Using arguements of C. Cosín and A. Ros, we can prove that there exists $(\Omega, \mathcal{Q}, \varphi)$ a multi-domain with logarithmic singularity such that M_+^* with the flat metric can be seen as the interior of Ω minus the singularity point $\mathcal{Q}$. Besides Ω is periodic because of the existence of the isometry t. The boundary of Ω is composed of half-lines. M_+^* is then a graph over Ω such that the line which is the conjugate of the circle of fixed points that contains no p_i is the part of the boundary of the graph which is above $\mathcal{Q}$ and the conjugates of the arcs $p_i p_{i+1}$ are lines over the vertices of Ω.

The quotient of Ω by the group $(t^n)_{n\in\mathbb{Z}}$ is then a multi-domain with cone singularity (D', Q, ψ). D' has r vertices $P_1, \dots, P_r$ and is bounded by $2r$ half-lines; more precisely, D' is a multi-domain with cone singularity D which is bounded by the flux polygon $F(M) = (v_1, \dots, v_r)$ (we have $v_i = \overrightarrow{\psi(P_i)\psi(P_{i+1})}$) to which we have glued r half-strips (along the edge $[P_i, P_{i+1}]$, we glue a half-strip isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+$). This proves that the flux polygon $F(M)$ bounds a multi-domain with cone singularity.

In Section 2 and Section 3, we shall prove many results that explains, when a polygon V bounds a multi-domain with cone singularity, how we can build a candidate for a surface M_+^* such that $F(M) = V$.

The main result we prove concerning the Plateau problem at infinity for genus one is the following.

Theorem 2. *Let $v_1, \dots, v_r$ be r non zero vectors of $\mathbb{R}^2$ such that $(v_1, \dots, v_r)$ is a polygon that bounds an immersed polygonal disk. Then there exists M*

an Alexandrov-embedded r-noid with genus one and horizontal ends of type I such that $F(M) = (v_1, \dots, v_r)$.

This implies that every polygon that can be realized as the flux polygon of an r-noid of genus 0 is also the flux polygon of an r-noid of genus 1.

2 A Dirichlet problem

Let $v_1, \dots, v_r$ be r non zero vectors of $\mathbb{R}^2$ such that $V = (v_1, \dots, v_r)$ bounds a multi-domain with cone singularity (D, Q, ψ). Let us note $P_1, \dots, P_r$ the vertices of the polygon, we put $P_{r+1} = P_1$; in the following, we suppose that the orientation on the polygon V is the one given as boundary of D (We remark that, in the following, the vertices of V will be identified with the vertices of D). Following Construction 1, we get a multi-domain with logarithmic singularity $(\mathcal{W}, \mathcal{Q}, \varphi)$ with a projection map $\pi : \mathcal{W} \to D$. Because of Remark 1, we suppose that the period $2q\pi$ of $\mathcal{W}$ is equal to the angle of the cone singularity of D, in fact we suppose that D is the quotient of $\mathcal{W}$ by the isometry f; In the following we shall say that D satisfies the hypothesis H.

Construction 3. Let $i \in \{1, \dots, r\}$ and consider E an edge of $\mathcal{W}$ which is send by π to the edge $[P_i, P_{i+1}]$ of D; we can glue to $\mathcal{W}$ along E a half strip S_i isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+^*$ (if $A \in E$ the point A is identified with $(\pi(A), 0)$). Making this for every i and every edges E, we get a new multi-domain with a logarithmic singularity at $\mathcal{Q}$ that we note $(\Omega, \mathcal{Q}, \varphi)$ (we keep the same notation for the developping map since it is an extension of the original developping map). Since $\mathcal{W}$ is periodic, Ω is periodic and have the same period, we note f the corresponding isometry.

Let us note, for $i \in \{1, \dots, r\}$, $\mathcal{L}_i^+$ (resp. $\mathcal{L}_i^-$) the union of the straight-lines corresponding to $\{P_i\} \times \mathbb{R}_+^*$ in the half-strips glued along the edges E such that $\pi(E) = [P_i, P_{i+1}]$ (resp. $\pi(E) = [P_{i-1}, P_i]$). $\mathcal{L}_i^+$ and $\mathcal{L}_i^-$ are a countable union of half straight-lines. We note $\mathcal{V}$ the set of the vertices of Ω this set is $\pi^{-1}\{P_1, \dots, P_r\}$.

We then have the following existence and uniqueness result.

Theorem 3. *Let (D, Q, ψ) be a multi-domain with cone singularity that bounds a polygon V, Constructions 1 and 3 give us a periodic multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$; we suppose that the period of Ω is the cone angle of D at Q. Then there exists a solution u of the minimal surface equation on Ω such that*

1. *u tends to $+\infty$ along $\mathcal{L}_i^+$ and $-\infty$ along $\mathcal{L}_i^-$ and*

2. $\Psi_u(\mathcal{Q}) = 0 = \Psi_u(\mathcal{V})$,

where $\mathcal{L}_i^+$, $\mathcal{L}_i^-$ and $\mathcal{V}$ are the notations given in Construction 3. Besides, the solution is unique up to an additive constant.

The conditions imposed to a function u solution of this Plateau problem make that the graph of u is a good candidate for giving a solution to the Plateau problem at infinity with V as flux polygon.

Corollary 1. *If u is a solution on Ω of the Dirichlet problem asked in Theorem 3, there exists a constant $c \in \mathbb{R}$ such that $u \circ f = u + c$.*

Proof. Let u be a solution of the Dirichlet problem asked in Theorem 3, then $u \circ f$ is also a solution of this Dirichlet problem. This proves that $u \circ f - u$ is constant. □

The following of this section is devoted to the proof of Theorem 3

2.1 Notations

First, the period of Ω is $2q\pi$. We can suppose that P_1 is such that $d(Q, P_1) = \min_i d(Q, P_i)$. Let us consider a vertex in $\mathcal{V}$ which is a lift of P_1 and denote it $\mathcal{P}_1(0)$. Since $d(M, P_1)$ is minimal, the geodesic joining $\mathcal{Q}$ to $\mathcal{P}_1(0)$ is embedded, this implies that the first coordinate of $\varphi(\mathcal{P}_1(0))$ is $d(\mathcal{Q}, \mathcal{P}_1(0)) = d(Q, P_1)$. By considering $R_\alpha \circ \varphi$ instead of φ, we can suppose that $\varphi(\mathcal{P}_1(0)) = (d(\mathcal{Q}, \mathcal{P}_1(0), 0)$. Let us note $\mathcal{P}_1(k) = f^k(\mathcal{P}_1(0))$ for $k \in \mathbb{Z}$; we then have $\varphi(\mathcal{P}_1(k)) = (d(M, P_0), 2kq\pi)$. Then $\{\mathcal{P}_1(i), i \in \mathbb{Z}\}$ is the set of the vertex of Ω corresponding to P_1 (*i.e.* $\{\mathcal{P}_1(i), i \in \mathbb{Z}\} = \pi^{-1}(P_1)$).

If we remove the geodesics $[\mathcal{Q}, \mathcal{P}_1(k)]$ and $[\mathcal{Q}, \mathcal{P}_1(l)]$ $(k < l)$ from Ω, we get a space which have three connected components; one of them is such that its intersection with $\mathcal{N}$, the neighborhood of $\mathcal{Q}$ introduced in Definition 3, is isometric with $]0, r[\times]2kq\pi, 2lq\pi[$ by φ; we shall note this part Ω_k^l. For $n \in \mathbb{Z}$ we have $f^n(\Omega_k^l) = \Omega_{k+n}^{l+n}$. In Ω_0^1, there is exactly one lift of every P_i $(2 \le i \le r)$, we note $\mathcal{P}_i(0)$ the lift of P_i that is in Ω_0^1. We then note for $k \in \mathbb{Z}$ $\mathcal{P}_i(k) = f^k(\mathcal{P}_i(0))$; we then have that $\mathcal{P}_i(k)$ is the only lift of P_i in Ω_k^{k+1}. A part Ω_k^{k+1} of Ω is called a *period* of Ω.

For $i \in \{1, \dots, r\}$ and $k \in \mathbb{Z}$, let us note $\mathcal{L}_i^+(k)$ (resp. $\mathcal{L}_i^-(k)$) the half straight-line included in $\mathcal{L}_i^+$ (resp. $\mathcal{L}_i^-$) and having $\mathcal{P}_i(k)$ as end-point.

Let us consider k and l in $\mathbb{Z}$ such that $k < l$, then Ω_k^l is a multi-domain. Its vertices are $\mathcal{Q}$, the $\mathcal{P}_1(m)$ for $k \le m \le l$ and the $\mathcal{P}_i(m)$ for $2 \le i \le r$ and $k \le m < l$. Its boundary is composed of two segments $[\mathcal{Q}, \mathcal{P}_1(k)]$ and $[\mathcal{Q}, \mathcal{P}_1(l)]$ and $2r(l-k)$ half straight-lines which are the $\mathcal{L}_1^+(m)$ for $k \le m < l$,

the $\mathcal{L}_1^-(m)$ for $k < m \le l$ and the $\mathcal{L}_i^\pm(m)$ for $2 \le i \le r$ and $k \le m < l$. From Ω_k^l, we now define a new multi-domain: we can glue to Ω_k^l two half-strips, one is isometric to $[\mathcal{Q}, \mathcal{P}_1(k)] \times \mathbb{R}_+$ and is glued along $[\mathcal{Q}, \mathcal{P}_1(k)]$, the second is isometric to $[\mathcal{P}_1(l), \mathcal{Q}] \times \mathbb{R}_+$ and is glued along $[\mathcal{P}_1(l), \mathcal{Q}]$. We note $\widetilde{\Omega}_k^l$ this new multi-domain. We note $\widetilde{\mathcal{L}}_k^-$ (resp. $\widetilde{\mathcal{L}}_l^+$) the new half straight-line in the boundary of $\widetilde{\Omega}_k^l$ with $\mathcal{P}_1(k)$ (resp. $\mathcal{P}_1(l)$) as end-point. We also note $\widetilde{\mathcal{L}}^+$ and $\widetilde{\mathcal{L}}^-$ the two half straight-lines in the boundary with $\mathcal{Q}$ as end-point such that $\widetilde{\mathcal{L}}^+$ is in the same half-strip than $\widetilde{\mathcal{L}}_k^-$.

2.2 Proof of the existence

We shall now prove the existence part of Theorem 3. First, for n in $\mathbb{N}^*$, we prove that there exists a function u_n on $\widetilde{\Omega}_{-n}^n$ which:

1. is a solution of (MSE),
2. tends to $+\infty$ on $\widetilde{\mathcal{L}}^+$, $\widetilde{\mathcal{L}}_n^+$ and all the $\mathcal{L}_i^+(k)$ that are in the boundary of $\widetilde{\Omega}_{-n}^n$ and
3. tends to $-\infty$ on $\widetilde{\mathcal{L}}^-$, $\widetilde{\mathcal{L}}_{-n}^-$ and all the $\mathcal{L}_i^-(k)$ that are in the boundary of $\widetilde{\Omega}_{-n}^n$.

Let us consider the following polygon:

$$(\overrightarrow{QP_1}, \underbrace{v_1, \dots, v_r, \dots\dots, v_1, \dots, v_r}_{2n \text{ times}}, \overrightarrow{P_1Q})$$

If we remove to $\widetilde{\Omega}_{-n}^n$ all the half-strips, we get a multi-domain which is bounded by the above polygon. Then, the existence of the solution u_n is ensured by Theorem 7 in [Ma1].

Let us now only consider the restriction of u_n to Ω_{-n}^n. We then have a sequence of solutions of (MSE) defined in an increasing sequence of domains Ω_{-n}^n and $\bigcup_{n\in\mathbb{N}^*} \Omega_{-n}^n = \Omega$. We can then consider that (u_n) is a sequence of functions on Ω. We want (u_n) to converge, so we shall prove that there is no line of divergence (see Appendix A).

First we give some preliminary results on u_n.

Lemma 2. *Let u be a solution of* (MSE) *on the half-strip $[0, a] \times \mathbb{R}_+$ such that u tends to $-\infty$ on $\{0\} \times \mathbb{R}_+$ and $+\infty$ on $\{a\} \times \mathbb{R}_+$. Then, for $y \ge 4a$,*

we have:

$$\frac{|p|}{W}(x,y) \geq 1 - \frac{a^2}{y^2}$$
$$\frac{|q|}{W}(x,y) \leq \sqrt{2}\frac{a}{y}$$

Proof. We note A the point of coordinates (x,y) and B a point in the boundary of the half-strip which realizes the distance from A to this boundary. Since $y \geq 4a$ the coordinates of B are $(0,y)$ or (a,y); besides the distance $|AB|$ is less than $a/2$. Because of the value of u on the boundary, the distance along the graph from the point above A to the boundary of the graph is bigger than $4a$. The ratio of this two distances is less than $1/8$, then we can apply Lemma 1 in [JS]; this gives the lemma. ☐

Corollary 2. *Let u be a solution of* (MSE) *on the half-strip* $[0,a]\times\mathbb{R}_+$ *such that u tends to $-\infty$ on $\{0\}\times\mathbb{R}_+$ and $+\infty$ on $\{a\}\times\mathbb{R}_+$. We have $\Psi_u(0,y) = \Psi_u(a,y)$ and if $\Psi_u(0,0) = 0$ then $\Psi_u \geq 0$ in the half-strip.*

Proof. Because of the value of u on the boundary we have $\Psi_u(0,y) = \Psi_u(0,0) + y$ and $\Psi_u(a,y) = \Psi_u(a,0) + y$. This implies that for $y \geq 4a$ we have

$$\begin{aligned}
|\Psi_u(0,0) - \Psi_u(a,0)| &= |\Psi_u(0,y) - \Psi_u(a,y)| \\
&= |\int_0^a -\frac{q}{w}(x,y)\mathrm{d}x| \\
&\leq \int_0^a \sqrt{2}\frac{a}{y}\mathrm{d}x \leq \sqrt{2}\frac{a^2}{y}
\end{aligned}$$

Then by letting y goes to $+\infty$ we have $\Psi_u(0,0) = \Psi_u(a,0)$ and then $\Psi_u(0,y) = \Psi_u(a,y)$.

If $\Psi_u(0,0) = 0$ then $\Psi_u(0,y) = y$. If A is in the interior of the half-strip, there exists y such that A is at a distance less than y from the point $(0,y)$; then, since Ψ_u is 1-Lipschitz continuous $\Psi_u(A) \geq 0$. ☐

This corollary implies that, if we choose Ψ_{u_n} such that $\Psi_{u_n}(\mathcal{Q}) = 0$, we have $\Psi_{u_n}(\mathcal{P}_i(k)) = 0$ for all $\mathcal{P}_i(k) \in \Omega^n_{-n}$ and $\Psi_{u_n} \geq 0$ in the half-trips contained in $\widetilde{\Omega}^n_{-n}$. Since Ψ_{u_n} satisfies a maximum principle, we have $\Psi_{u_n} \geq 0$ in Ω^n_{-n}.

Assume that there exists a line of divergence L. By Lemma A.1, L can not have an end point in the interior of one of the half straight-lines that

compose the boundary of Ω; this prove that if it has end-points it must be $\mathcal{Q}$ or one $\mathcal{P}_i(k)$.

If L has no end point, let $\mathcal{A}$ be a point of L and let us note d the distance between $\mathcal{A}$ and $\mathcal{Q}$, since $\Psi_{u_n}(\mathcal{Q}) = 0$ and Ψ_{u_n} is 1-Lipschitz, we have $0 \leq \Psi_{u_n}(\mathcal{A}) \leq d$. Let us fix an orientation to L such that the limit normal along this line of divergence is the right-hand unit normal. We then note s the arc-length along L with $\mathcal{A}$ as origin. Let us consider $\mathcal{A}'$ the point on L of arc-length $s = -2d$. Then we know that, for the subsequence that makes L appear, we have $\Psi_{u_{n'}}(\mathcal{A}) - \Psi_{u_{n'}}(\mathcal{A}') \to 2d$ but, since $\Psi_{u_n}(\mathcal{Q}) \leq d$ this implies that $\Psi_{u_{n'}}(\mathcal{A}') < 0$ for big n'; this is a contradiction.

If L is a segment (L has two end-points noted $\mathcal{A}_1$ and $\mathcal{A}_2$), we know that for each n we have $\Psi_{u_n}(\mathcal{A}_1) = 0 = \Psi_{u_n}(\mathcal{A}_2)$, but for the subsequence that make L appear we have $|\Psi_{u_{n'}}(\mathcal{A}_1) - \Psi_{u_{n'}}(\mathcal{A}_2)| \to d(\mathcal{A}_1, \mathcal{A}_2) > 0$; this gives us a contradiction.

We then can suppose that L has only one end-point that we note $\mathcal{F}$ and goes to infinity in one half-strip that we can parametrized isometricaly by $[0,a] \times \mathbb{R}_+$. We remark that for one half-strip in Ω the number of such lines L is finite. We have, for each n, $\Psi_{u_n}(\mathcal{F}) = 0$. There exists $b \in]0, a[$ such that the part of L in the half-strip is $\{b\} \times \mathbb{R}_+$. By changing L if necessary we can suppose that the part $]b, a[\times \mathbb{R}_+$ is included in $\mathcal{B}(u_n)$. Let $\mathcal{A} = (b, 0)$, $\mathcal{B} = (b, 2(a-b))$, $\mathcal{C} = (a, 2(a-b))$ and $\mathcal{D} = (a, 0)$ be four points in the half-strip. Since $\mathrm{d}\Psi_{u_n}$ is closed we have:

$$\begin{aligned}
\int_{[\mathcal{A},\mathcal{B}]} \mathrm{d}\Psi_{u_n} &= -\int_{[\mathcal{B},\mathcal{C}]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{C},\mathcal{D}]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{D},\mathcal{A}]} \mathrm{d}\Psi_{u_n} \\
&= 2(a-b) - \int_{[\mathcal{B},\mathcal{C}]} \mathrm{d}\Psi_{u_n} - \int_{[\mathcal{D},\mathcal{A}]} \mathrm{d}\Psi_{u_n} \\
&\geq 2(a-b) - (a-b) - (a-b) = 0
\end{aligned}$$

This implies that we have only one possibility for the limit normal . Since $]b, a[\times \mathbb{R}_+ \subset \mathcal{B}(u_n)$ we can suppose that a subsequence $(u_{n'})$ converges to a function v on $]b, a[\times \mathbb{R}_+$ (n' is chosen such that the line of divergence L appears). v is a solution of (MSE) and by Lemma A.2, we know that v tends to $+\infty$ along $\{b\} \times \mathbb{R}_+^*$ and $-\infty$ along $\{a\} \times \mathbb{R}_+^*$. Then by Lemma 2, $\Psi_v(\mathcal{A}) = \Psi_v(\mathcal{D}) = \lim \Psi_{u_n}(\mathcal{D}) = 0$. But, for the subsequence, $\Psi_{u_{n'}}(\mathcal{A}) - \Psi_{u_{n'}}(\mathcal{F}) \to d(\mathcal{F}, \mathcal{A}) > 0$, this contradicts the fact that $\Psi_{u_n}(\mathcal{F}) = 0$ and $\lim \Psi_{u_{n'}}(\mathcal{A}) = 0$.

We then have prove that $\mathcal{B}(u_n) = \Omega$, and, by taking a subsequence, we can suppose that u_n converges to a solution u of (MSE) on Ω. By Lemma

A.2, u is such that u tends to $+\infty$ along the $\mathcal{L}_i^+(k)$ and $-\infty$ along the $\mathcal{L}_i^-(k)$. By construction, we have also that $\Psi_u(\mathcal{Q}) = 0 = \Psi_u(\mathcal{P}_i(k))$ and $\Psi_u \geq 0$.

2.3 Property of the solution 1

Before proving the uniqueness, we need to give some properties of a solution of the Dirichlet problem asked in Theorem 3. We recall that there exist ρ_0 and a neighborhood $\mathcal{N}$ of $\mathcal{Q}$ such that φ is an isometry from $\mathcal{N}$ into $\{M \in \mathcal{D} | d(\mathcal{O}, M) \leq \rho_0\}$. We then can use the polar coordinates for v a function defined on $\mathcal{N}$.

Proposition 2. *Let ε be a positive number. There exists $d > 0$ such that for every $\alpha \in \mathbb{R}$ and every v solution of* (MSE) *on $\mathcal{N} \cap \{\alpha - \pi < \theta < \alpha + \pi\}$:*

$$\sup_{0<\rho\leq\rho_0} v(\rho,\alpha) < \sup_{[d,\rho_0]\times[\alpha-3\pi/4,\alpha+3\pi/4]} v(\rho,\theta) + \varepsilon$$
$$\inf_{0<\rho\leq\rho_0} v(\rho,\alpha) > \inf_{[d,\rho_0]\times[\alpha-3\pi/4,\alpha+3\pi/4]} v(\rho,\theta) - \varepsilon$$

The proof is based on the idea used by R. Osserman to prove Theorem 10.3 in [Os].

Proof. First let us assume that $\alpha = \pi/2$. Then $\mathcal{N} \cap \{\alpha - \pi < \theta < \alpha + \pi\}$ is isometric to the disk of center the origin and radius ρ_0 minus the segment joining the origin to $(0, -\rho_0)$. Then we only prove the first inequality since the second is a consequence of the first by substituting v for $-v$.

Let $\varepsilon > 0$ and d be a positive number, we note A the point of coordinates $(0, -d)$ and D the domain in $D(0, \rho_0)$ delimited by the segment joining $(-\sqrt{\rho_0^2 - d^2}, -d)$ to $(-d, -d)$, the half circle of center A and radius d: $\theta \mapsto (d\cos\theta, d(\sin\theta - 1))$ for $\theta \in [0, \pi]$ and the segment joining $(d, -d)$ to $(\sqrt{\rho_0^2 - d^2}, -d)$ and containing $(0, \rho_0/2)$. On the domain D we consider the function $c : M \mapsto d\left(-\operatorname{argch}(\frac{|AM|}{d}) + \operatorname{argch}(\frac{\rho_0+d}{d})\right)$. The graph of c is a piece of a catenoid and c is a positive function upper bounded by $d\operatorname{argch}(\frac{\rho_0+d}{d})$. Let d be small enough such that this upper-bound is less than ε.

On the part of the boundary of D which is a half circle of center A, the derivatives $\nabla c \cdot n$, where n is the outward pointing normal, takes the value $+\infty$. This implies, by Lemma 10.2 in [Os], that $c + K \geq v$ where K is the supremum of v on the boundary of D minus the half circle of center A. Since this part of the boundary is included in the set of points of polar coordinates included in $[d, \rho_0] \times [-\pi/4, 5\pi/4]$ and D contains the segment joining the origin to $(0, 1)$ the proposition is established. □

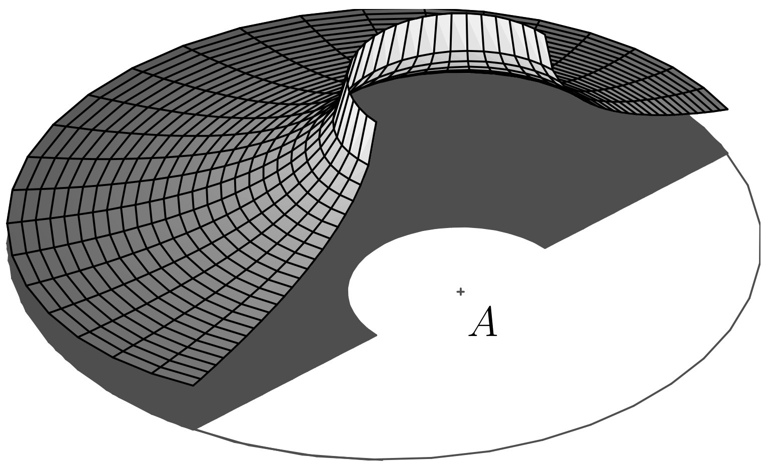

Figure 1: the graph of the function c

This proposition is an important fact since it implies that if u is a solution of the dirichlet problem asked in Theorem 3, u is bounded on $\mathcal{N} \cap \Omega_k^l$ for every k and l.

2.4 Proof of the uniqueness

Let us now prove the uniqueness part of Theorem 3. We consider u and v two solutions of the Dirichlet problem asked in Theorem 3 and we assume that these two solutions are different: $u - v$ is not constant. By changing v by $v + c$ where c is a real constant we can suppose:

1. $\{u - v > 0\} \cap \Omega_0^{+\infty}$ and $\{u - v < 0\} \cap \Omega_0^{+\infty}$ are non-empty ($\Omega_0^{+\infty}$ is $\bigcup\limits_{n>0} \Omega_0^n$) and
2. the segment $[\mathcal{Q}, \mathcal{P}_1(0)]$ is included in $\{u - v > 0\}$.

The first assertion is due to the fact that $u - v$ is non-constant and that we can exchange u and v, the second assertion is a consequence of the fact that u and v are bounded on the segment $[\mathcal{Q}, \mathcal{P}_1(0)]$ by Proposition 2 and Lemma 2 in [Ma1].

We note $\Delta = \{u - v < 0\} \cap \Omega_0^{+\infty}$ We then note Δ^n the intersection of Δ and Ω_0^n. The boundary $\partial\Delta^n$ is composed of a part included in the boundary of Ω, a part included in the interior of Ω_0^n and a part included in the segment $[\mathcal{Q}, \mathcal{P}_1(n)]$. Along the first part $\mathrm{d}\widetilde{\Psi} = \mathrm{d}\Psi_u - \mathrm{d}\Psi_v = 0$ and along the second

part $\mathrm{d}\widetilde{\Psi}$ is positive by Lemma 2 in [CK]; the union of this first two parts will be noted $\partial^n\Delta$.

We note Δ_n^l the intersection of Δ_n with the part of Ω_0^n such that a point A is in this part if either A is not in one half-strip included in Ω_0^n or $A=(x,y)$ is in one of these half-strips which is isometricaly parametrized by $[0,a]\times\mathbb{R}_+$ and $y\leq l$. The boundary of Δ_n^l is composed of three parts: the first is $\partial\Delta_n\cap[\mathcal{Q},\mathcal{P}_1(n)]$, the second is $\partial^n\Delta\cap\overline{\Delta_n^l}$ and the third is included in the union of the segments parametrized by $[0,a]\times\{l\}$ in each half-strip; we note Γ^l this third part. Since $\mathrm{d}\widetilde{\Psi}$ is closed we have:

$$\int_{\partial\Delta_n\cap[\mathcal{Q},\mathcal{P}_1(n)]}\mathrm{d}\widetilde{\Psi}+\int_{\partial^n\Delta\cap\overline{\Delta_n^l}}\mathrm{d}\widetilde{\Psi}+\int_{\Gamma^l}\mathrm{d}\widetilde{\Psi}=0 \tag{2}$$

We then have

$$\int_{\partial^n\Delta\cap\overline{\Delta_n^l}}\mathrm{d}\widetilde{\Psi}\leq|\mathcal{Q}\mathcal{P}_1(n)|+\left|\int_{\Gamma^l}\mathrm{d}\widetilde{\Psi}\right|$$

Since $\mathrm{d}\widetilde{\Psi}\geq 0$ along $\partial^n\Delta$, $\int_{\partial^n\Delta\cap\overline{\Delta_n^l}}\mathrm{d}\widetilde{\Psi}$ increases as l increases. By Lemma 2, $\int_{\Gamma^l}\mathrm{d}\widetilde{\Psi}\to 0$ as l goes to $+\infty$. Then the above equation implies that $\mathrm{d}\widetilde{\Psi}$ is integrable on $\partial^n\Delta$ and by passing to the limit in (2), we obtain

$$\int_{\partial\Delta_n\cap[\mathcal{Q},\mathcal{P}_1(n)]}\mathrm{d}\widetilde{\Psi}+\int_{\partial^n\Delta}\mathrm{d}\widetilde{\Psi}=0 \tag{3}$$

This equation implies that

$$\int_{\partial^n\Delta}\mathrm{d}\widetilde{\Psi}\leq|\mathcal{Q}\mathcal{P}_1(n)|=|\mathcal{Q}\mathcal{P}_1(0)|$$

Then $\mathrm{d}\widetilde{\Psi}$ is integrable on $\partial\Delta$ and $\int_{\partial\Delta}\mathrm{d}\widetilde{\Psi}>0$ since the part of $\partial\Delta$ included in the interior of Ω is non empty. We have

$$\int_{\partial\Delta}\mathrm{d}\widetilde{\Psi}=\sum_{n=0}^{+\infty}\int_{\partial\Delta\cap\Omega_n^{n+1}}\mathrm{d}\widetilde{\Psi}$$

Then $\displaystyle\int_{\partial\Delta\cap\Omega_n^{n+1}}\mathrm{d}\widetilde{\Psi}\longrightarrow 0$. We then want to understand what happens on Ω_n^{n+1} for big n.

Let us consider, for $n\in\mathbb{N}$, $u_n=u\circ f^{-n}$ and $v_n=v\circ f^{-n}$; the restriction of u_n to Ω_0^1 is equal to the restriction of u to Ω_n^{n+1} and the same is true for v. With the same arguements as in the proof of existence, we can prove that there exist two real sequences (a_n) and (b_n) such that $u_{n'}-a_{n'}\to\tilde{u}$ and

$v_{n'} - b_{n'} \to \tilde{v}$ on Ω (in fact $a_n = u_n(P)$ and $b_n = v_n(P)$ for a point P in Ω). The functions $\tilde{u}$ and $\tilde{v}$ are two solutions of the Dirichlet problem asked in Theorem 3 and the convergence is uniform for every derivative on compact parts. By changing our subsequence if necessary, we can assume that $b_{n'} - a_{n'} \to \pm\infty$ or $b_{n'} - a_{n'} \to c \in \mathbb{R}$. We are interested in $\{u_n - v_n < 0\} \cap \Omega_0^1$ and, in a certain sense, these sets "converge" to $\{\tilde{u} - \tilde{v} < \lim b_{n'} - a_{n'}\} \cap \Omega_0^1$.

Let $\gamma : [0, |\mathcal{Q}\mathcal{P}_1(1)|] \to [\mathcal{Q}, \mathcal{P}_1(1)]$ the parametrization by arc-length of the segment $[\mathcal{Q}, \mathcal{P}_1(1)]$, with $\gamma(0) = \mathcal{Q}$.

First we assume that $b_{n'} - a_{n'} \to \pm\infty$. Let $\varepsilon > 0$ such that ε is less than $\dfrac{1}{3}\displaystyle\int_{\partial^n\Delta} \mathrm{d}\widetilde{\Psi}$ and $\dfrac{1}{2}|\mathcal{Q}, \mathcal{P}_1(1)|$. Since $\tilde{u}$ and $\tilde{v}$ are bounded on $[\mathcal{Q}, \mathcal{P}_1(1)]$ and that we have uniform convergence on $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$, we can ensure that, for n' big enough, $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ is included in $\{u_{n'} - v_{n'} < 0\}$ or does not intersect $\{u_{n'} - v_{n'} < 0\}$ following the sign of the limit of $b_{n'} - a_{n'}$. For every n, $\displaystyle\int_{[\mathcal{Q}, \mathcal{P}_1(n)]} \mathrm{d}\widetilde{\Psi} = 0$ because of our hypotheses on Ψ_u and Ψ_v, we then have for big n'

$$\left| \int_{\partial\Delta_{n'+1} \cap [\mathcal{Q}, \mathcal{P}_1(n'+1)]} \mathrm{d}\widetilde{\Psi} \right| \leq 2\varepsilon \tag{4}$$

This equality implies that $\displaystyle\int_{\partial^{n'+1}\Delta} \mathrm{d}\widetilde{\Psi} \leq 2\varepsilon$ for big n'. Then by passing to the limit we obtain a contradiction because of our hypothesis on ε.

We assume now that $b_{n'} - a_{n'} \to c$. Let ε be as above. If the segment $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ is included in $\{\tilde{u} - \tilde{v} < c\}$ or does not intersect $\{\tilde{u} - \tilde{v} < c\}$ then the same property is true for $\{u_{n'} - v_{n'} < 0\}$ for big n. Then same arguements as above give us a contradiction. We then can ensure that there is a point in $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$ where $\tilde{u} - \tilde{v} = c$. If $\tilde{u} = \tilde{v} + c$ on Ω, we have $\mathrm{d}\Psi_{u_{n'}} - \mathrm{d}\Psi_{v_{n'}} \to 0$ uniformly on $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$. Then for n' big enough, the inequality (4) is true, this gives us a countradiction. Then $\tilde{u} \neq \tilde{v} + c$ on Ω and there is a compact part of the boundary of $\{\tilde{u} - \tilde{v} < c\}$ that crosses the segment $\gamma([\varepsilon, |\mathcal{Q}\mathcal{P}_1(1)| - \varepsilon])$, let us note Γ this part of the boundary (Γ is oriented as the boundary of this set). We have $\displaystyle\int_\Gamma \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}} > 0$ by Lemma 2 in [CK]. The curve Γ is included in a compact part of Ω_0^2 and since $(u_{n'} - v_{n'})$ converges to $\tilde{u} - \tilde{v} - c$ (the convergence is uniform for each derivative on every compact part) we can ensure that for n' big enough

$$\int_{\partial\{u_{n'} - v_{n'} < 0\} \cap \Omega_0^2} \mathrm{d}\Psi_{u_{n'}} - \mathrm{d}\Psi_{v_{n'}} \geq \frac{1}{2}\int_\Gamma \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}}$$

This implies that

$$\int_{\partial\Delta\cap\Omega_{n'}^{n'+2}} \mathrm{d}\widetilde{\Psi} \geq \frac{1}{2}\int_{\Gamma} \mathrm{d}\Psi_{\tilde{u}} - \mathrm{d}\Psi_{\tilde{v}} > 0$$

We have then a contradiction with $\displaystyle\int_{\partial\Delta\cap\Omega_n^{n+1}} \mathrm{d}\widetilde{\Psi} \longrightarrow 0$. We have then proved that our two solutions u and v differ only by a real additive constant.

2.5 Property of the solution 2

From Corollary 1, if u is a solution of the Dirichlet problem asked in Theorem 3, there exists a constant $c \in \mathbb{R}$ such that $u \circ f = u + c$. We have the following result for such a situation.

Proposition 3. *Let $(\Omega, \mathcal{Q}, \varphi)$ be a periodic multi-domain with logarithmic singularity, we note f the isometry associated to the periodicity. Let v be a solution of the minimal surface equation on Ω such that:*

1. *there exists a constant $c \in \mathbb{R}$ such that $v \circ f = v + c$ and*
2. *if Ψ_v is the conjugate function to v with $\Psi_v(\mathcal{Q}) = 0$, Ψ_v is non-negative.*

Under these hypotheses, the constant c is non zero.

Proof. Let us suppose that the constant c is zero and consider v only on the neighborhood $\mathcal{N}$ of the singularity point $\mathcal{Q}$. Since $v \circ f = v$, by taking the quotient with respect to f, the function v can be seen as a function $\widetilde{v}$ defined on $\mathcal{C}$ a flat disk of radius ρ_0 with a cone singularity at its center of angle $2q\pi$ minus the singularity point (i.e. $\mathcal{C}$ is $\{(\rho,\theta), \rho \in (0,\rho_0), \theta \in [0, 2q\pi]\}$ where $(\rho, 0)$ and $(\rho, 2q\pi)$ are identified, the metric on $\mathcal{C}$ is the polar metric). The graph of $\widetilde{v}$ is a minimal surface of $\mathcal{C} \times \mathbb{R}$. This surface is topologicaly an annulus then it is conformally parametrized by a Riemann surface R which is an annulus: there are two harmonic maps $X : R \to \mathcal{C}$ and $x_3 : R \to \mathbb{R}$ such that $x_3 = \widetilde{v} \circ X$. We can suppose that R is either $\{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$ for $1 < a \leq +\infty$ or $\{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$ for $0 < a \leq +\infty$. On $\mathcal{C}$, the function $\Psi_{\widetilde{v}}$ is well defined and satisfies $\Psi_{\widetilde{v}} \geq 0$ and $\Psi_{\widetilde{v}} = 0$ at the cone singularity by hypothesis. Since x_3 is harmonic, on R we can define its harmonic conjugate x_3^*; *a priori* x_3^* is multi-valuated but, for a good choice of x_3^*, we have $x_3^* = \Psi_{\widetilde{v}} \circ X$ then x_3^* is well defined on R. As $\Psi_{\widetilde{v}} = 0$ at the cone singularity, $x_3^*(\zeta)$ converges to zero as either $|\zeta|$ tends to 1 when $R = \{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$ or $|\zeta|$ tends to zero when $R = \{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$.

In the second case ($R = \{\zeta \in \mathbb{C} | 0 < |\zeta| < a\}$), x_3^* is a harmonic function on a pointed disk, which has a continuous extension to the whole disk. But, an harmonic function can not have an isolated singularity, then the continuous extension is harmonic and have an extremum at the origin, since $x_3^* \geq 0$ on R; this gives a contradiction.

In the other case ($R = \{\zeta \in \mathbb{C} | 1 < |\zeta| < a\}$), since $x_3^* = 0$ on the unit circle, the function x_3^* extends to an harmonic function on $\{\zeta \in \mathbb{C} | \frac{1}{a} < |\zeta| < a\}$ by Schwartz reflection principle (the extension is defined by $x_3^*(\zeta) = -x_3^*(\frac{1}{\bar{\zeta}})$; see [ABR]). By taking the harmonic conjugate, x_3 also extends to $\{\zeta \in \mathbb{C} | \frac{1}{a} < |\zeta| < a\}$. Along the unit circle $\mathbb{S}^1$, $\nabla x_3^* \cdot n = 0$, with n the unit tangent vector to $\mathbb{S}^1$, since $x_3^* = 0$ along this circle. Besides by Rolle's Theorem there is a point on $\mathbb{S}^1$ where $\nabla x_3 \cdot n = 0$. At this point, $\nabla x_3^* = 0$ and the local structure of a critical point of a non constant harmonic function implies that x_3^* must be negative on R in the neighborhood of this point, this contradicts $x_3^* \geq 0$ on R. $\square$

Remark 2. This proposition proves that, for a solution u of the Dirichlet problem asked in Theorem 3, the constant c such that $u \circ f = u + c$ is non-zero, then u can not pass to the quotient by f. But since $u \circ f = u + c$ the derivatives of u are invariant by f then the derivatives of u are well defined on the quotient. Besides, this implies that the function Ψ_u is also invariant by f.

An other property of such a solution u is that u can be build by the way used in the proof of the existence. As a consequence, the conjugate function Ψ_u is non-negative on Ω.

3 The regularity over the singularity point

As in the preceding section we consider V a polygon that bounds a multi-domain with cone singularity (D, Q, ψ); using Construction 1 and 3, we get a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$; we suppose that D satisfies the hypothesis H. Theorem 3 allows us to construct a minimal surface in $\mathbb{R}^3$ which is a graph over Ω; we want to use this surface to build a r-noid with genus 1 and horizontal ends of type I with V as flux polygon therefore the graph needs to be regular up to its boundary. In this section, we understand the behaviour of this minimal surface over the singularity point $\mathcal{Q}$: we shall prove the following result.

Theorem 4. *Let u be a solution of the Dirichlet problem asked in Theorem 3. Then the surface $\{\varphi_{\psi(Q)}(x), u(x)\}$ for $x \in \mathcal{N}$ is a minimal surface with*

boundary and its boundary is $\{\varphi_{\psi(Q)}(x), u(x)\}$ for $x \in \partial\mathcal{N}$ and the vertical straight line passing through $\psi(Q)$.

In fact, this surface can also be seen as a minimal surface in $\mathcal{D} \times \mathbb{R}$; with this point of view, Theorem 4 can be stated as follows.

Theorem 5. *Let u be a solution of the Dirichlet problem asked in Theorem 3. Then the surface $\{\varphi(x), u(x)\}_{x\in\mathcal{N}} \subset \mathcal{D} \times \mathbb{R}$ is a minimal surface with boundary and its boundary is $\{\varphi(x), u(x)\}$ for $x \in \partial\mathcal{N}$ and the vertical straight line $\{\mathcal{O}\} \times \mathbb{R}$.*

It is easy to see that Theorem 5 implies Theorem 4, then we shall prove Theorem 5. The problem in this theorem is the behaviour near the singularity point not along $\partial\mathcal{N}$.

3.1 A compactness result

In the proof of Theorem 5, we shall need to make converge a sequence of minimal surfaces. The following theorem will be an important tool.

Theorem 6. *Let ζ_1, ζ_2 and ζ_3 be three different points in $\mathbb{S}^1$. Let $\gamma_n : \mathbb{S}^1 \to \mathcal{D} \times \mathbb{R}^{k_1}$ be a sequence of Jordan curves in $\mathcal{D} \times \mathbb{R}^{k_1}$ and $f_n : \mathbb{S}^1 \to \mathbb{R}^{k_2}$ be a sequence of continuous maps; we note $g_n = (\gamma_n, f_n)$. We note $\Gamma_n = \gamma_n(\mathbb{S}^1)$ and put, for $1 \le i \le 3$, $p_i^n = \gamma_n(\zeta_i)$. We suppose that:*

1. *$I = \inf_{n\in\mathbb{N}}\{d(p_1^n, p_2^n), d(p_1^n, p_3^n), d(p_2^n, p_3^n)\} > 0$,*
2. *for every $m > 0$ there exists $\varepsilon > 0$ such that, for every $n \in \mathbb{N}$, if $\zeta, \zeta' \in \mathbb{S}^1$ and $d(g_n(\zeta), g_n(\zeta')) < \varepsilon$, one component of $\Gamma_n \backslash \{\gamma_n(\zeta), \gamma_n(\zeta')\}$ is of diameter less than m and*
3. *there exist $M > 0$ and a sequence $X_n : \Delta \to \mathcal{D} \times \mathbb{R}^{k_1+k_2}$ (Δ is the unit disk) such that $g_n = X_n|_{\mathbb{S}^1}$ and*
$$\int_\Delta (|X_{nx}|^2 + |X_{ny}|^2)\mathrm{d}x\mathrm{d}y < M$$

The family $\{\gamma_n\}_{n\in\mathbb{N}}$ is then equicontinuous.

By Arzela's theorem, this implies that if, for example, the sequence (p_1^n) converges there exists a subsequence $\gamma_{n'}$ that converges for uniform convergence. Theorem 6 is in fact very similar to a classical result used in the resolution of the classical Plateau problem (see for example, Lemma 3.2 in [Cou] or [Hi]).

Proof. The proof is based on the following lemma.

Lemma 3 (Courant-Lebesgue). *Let X be of class $C^0(\Delta, \mathcal{D} \times \mathbb{R}^k) \cap C^1(\mathring{\Delta}, \mathcal{D} \times \mathbb{R}^k)$ ($\mathring{\Delta}$ is the interior of Δ) and satisfy*

$$\int_{\Delta} (|X_x|^2 + |X_y|^2) \mathrm{d}x \mathrm{d}y < M$$

for some $M \in \mathbb{R}_+$. Then, for every $\zeta_0 \in \mathbb{S}^1$ and for each $\delta \in (0,1)$, there exists a number $\rho \in (\delta, \sqrt{\delta})$ such that the distance between the images $X(\zeta)$, $X(\zeta')$ of the two intersection points ζ and ζ' of $\mathbb{S}^1$ with the circle $\partial D_\rho(\zeta_0)$ can be estimated by

$$d(X(\zeta), X(\zeta')) \leq \left(\frac{4M\pi}{\ln(1/\delta)} \right)^{1/2}$$

The proof of this lemma can be found in [Hi].

Let e be a positive number, we suppose that $e < I$. By the second hypothesis, there exists ε such that, for every $n \in \mathbb{N}$, if $\zeta, \zeta' \in \mathbb{S}^1$ and $d(g_n(\zeta), g_n(\zeta')) < \varepsilon$, one component of $\Gamma_n \backslash \{\gamma_n(\zeta), \gamma_n(\zeta')\}$ is of diameter less than e. Let $\delta > 0$ be such that

$$\left(\frac{4M\pi}{\ln(1/\delta)} \right)^{1/2} \leq \varepsilon$$

and for every $\zeta \in \mathbb{S}^1$ we have $|\zeta - \zeta_i| > \delta$ for at least two of the points $\zeta_1, \zeta_2, \zeta_3$.

Let $n \in \mathbb{N}$ and $\zeta_0 \in \mathbb{S}^1$, by the third hypothesis and the Courant-Lebesgue Lemma, there exists $\delta < \rho < \sqrt{\delta}$ such that, if ζ and ζ' are the two intersections points of $\mathbb{S}^1$ with the circle $\partial D_\rho(\zeta_0)$, $d(X_n(\zeta), X_n(\zeta')) \leq \varepsilon$. $\mathbb{S}^1 \backslash \{\zeta, \zeta'\}$ is composed of two arcs: one, A', contains ζ_0 and all the points that are at a distance less than δ from ζ_0, the second arc, A'', contains two of the three points $\zeta_1, \zeta_2, \zeta_3$. Since $d(X_n(\zeta), X_n(\zeta')) \leq \varepsilon$, one of the two arcs $\gamma_n(A')$, $\gamma_n(A'')$ is of diameter less than e; but $\gamma_n(A'')$ contains two of the points p_1^n, p_2^n, p_3^n and $e < I$, then it is $\gamma_n(A')$ that is of diameter less than e. We then have proved that if $|\zeta - \zeta_0| < \delta$, $d(\gamma_n(\zeta), \gamma_n(\zeta_0)) \leq e$; this proves that the family $\{\gamma_n\}$ is equicontinuous. □

3.2 Proof of Theorem 5

3.2.1 Preliminaries

Let u be a solution of the Dirichlet problem asked in Theorem 3. We use the notations introduced in Section 2. The function u is contructed as the limit

of a sequence (u_n) where u_n is a solution of a Dirichlet problem on $\widetilde{\Omega}^n_{-n}$. Since we are only interested in the behaviour on $\mathcal{N}$ we shall use the polar coordinates given by φ; u is then defined on $[0,\rho_0]\times\mathbb{R}$ and u_n is defined on $[0,\rho_0]\times[-2nq\pi-\frac{\pi}{2},2nq\pi+\frac{\pi}{2}]$. Because of the periodicity of u, to prove Theorem 5, it is enough to make the proof on a period $[0,\rho_0]\times[0,2q\pi]$.

By construction, u_n takes the value $+\infty$ on $(0,\rho_0)\times\{-2nq\pi-\frac{\pi}{2}\}$ and the value $-\infty$ on $(0,\rho_0)\times\{2nq\pi+\frac{\pi}{2}\}$, besides, by taking $\Psi_{u_n}(\mathcal{Q})=0$, we have $\Psi_{u_n}\geq 0$ on $\widetilde{\Omega}^n_{-n}$, this proves, by Theorem 3 in [Ma1], that the minimal surface $\{\varphi(x),u_n(x)\}_{x\in\mathcal{N}\cap\widetilde{\Omega}^n_{-n}}\subset\mathcal{D}\times\mathbb{R}$ has the vertical straight-line passing by $\mathcal{O}$ as boundary. The idea of the proof of Theorem 5 is to follow the behaviour of the graph near this vertical straight-line when n goes to $+\infty$.

We now need a result on the behaviour of graph bounded by vertical line.

Lemma 4. *Let v be a solution of* (MSE) *on a sector of $\mathcal{D}$ $\{(r,\theta)\in\mathcal{D}|\ r\leq r_0,\ \alpha_1\leq\theta\leq\alpha_2\}$ $(\alpha_1<0<\alpha_2)$. Suppose that the graph of v in $\mathbb{R}^3$ is a complete minimal surface with boundary and the part of the boundary over the origin is an interval of the vertical straight-line passing by the origin. Then if v is bounded on $\{\theta=0\}$, $\lim\limits_{r\to0}v(r,0)$ exists. Besides the normal to the graph v at the points $(0,0,\lim\limits_{r\to0}v(r,0))$ is $\pm(0,1,0)$.*

Proof. We note Σ the graph of v. All the cluster points of $(r,0,v(r,0))$ as r goes to 0 are in the boundary of the graph and more precisely in the part of the boundary consisting in the interval of the vertical straight-line passing by the origin. Besides, the curve $r\mapsto(r,0,v(r,0))$ is in the intersection of the vertical plane $\{y=0\}$ and the graph of v, this curve is then in the intersection of two minimal surfaces. Let $(0,0,a)$ be a cluster point of $(r,0,v(r,0))$ as r goes to 0, since the boundary of Σ is a vertical straight-line near $(0,0,a)$ the normal to Σ at this point is horizontal and is $(\cos\alpha,\sin\alpha,0)$ for some $-\pi\leq\alpha<\pi$. Near $(0,0,a)$, Σ is then a graph over the vertical plane $\{x\cos\alpha+y\sin\alpha=0\}$ and is tangent to this plane at $(0,0,a)$. Then, if $|\alpha|\neq\pi/2$, the intersection of Σ and $\{y=0\}$ is only the vertical straight-line near $(0,0,a)$ then no $(r_n,0,v(r_n,0))$ can converge to $(0,0,a)$. This implies that the normal at $(0,0,a)$ is $\pm(0,1,0)$ and then since Σ is a graph over $\{y=0\}$ the intersection of Σ with this plane near $(0,0,a)$ is the vertical straight-line and some smooth curves passing by $(0,0,a)$ such that $x\neq0$ along them. One of this curve is then $(r,0,v(r,0))$, by continuity; this prove that $\lim\limits_{r\to0}v(r,0)=a$. ☐

We want to understand the convergence of the graph of u_n near the

singularity point $\mathcal{O}$. let $\alpha \in \mathbb{R}$, to simplify our notation in the following we suppose that $\alpha \in [0, 2q\pi]$, let $k \in \mathbb{N}$ and note $\alpha_1 = \alpha - 2kq\pi$ and $\alpha_2 = \alpha + 2kq\pi$. We know that $u_n(r, \alpha_1)$ and $u_n(r, \alpha_2)$ are bounded by Lemma 2 in [Ma1]. Then since the graph of u_n has a vertical straight-line as boundary, $u_n(r, \alpha_1) \to a_n$, $u_n(r, \alpha_2) \to b_n$ and $u_n(r, \alpha) \to c_n$ as r goes to 0, for some $c_n \in (b_n, a_n)$. The graph $\Sigma_n \subset \mathcal{D} \times \mathbb{R}$ of u_n over $U = [0, \rho_0] \times [\alpha_1, \alpha_2]$ is a minimal surface bounded by $\rho \mapsto (\rho, \alpha_1, u_n(\rho, \alpha_1))$ for $0 \leq \rho \leq \rho_0$, $\theta \mapsto (\rho_0, \theta, u_n(\rho_0, \theta))$ for $\alpha_1 \leq \theta \leq \alpha_2$, $\rho \mapsto (\rho, \alpha_2, u_n(\rho, \alpha_2))$ for $\rho_0 \geq \rho \geq 0$ and the segment $\{\mathcal{O}\} \times [b_n, a_n]$. Σ_n is a minimal surface of $\mathcal{D} \times \mathbb{R}$ which is of the type of the disk then we can parametrize conformally it by the upper half-disk $\Delta^+ = \{\zeta \in \Delta | \Im(\zeta) \geq 0\}$: there exist an harmonic map $X_n : \Delta^+ \to \mathcal{D}$ and an harmonic function $z_n : \Delta^+ \to \mathbb{R}$ such that the surface Σ_n is $\{(X_n(\zeta), z_n(\zeta)), \zeta \in \Delta^+\}$. We choose X_n and z_n such that $(X_n(-1), z_n(-1)) = (0, 0, a_n)$, $(X_n(0), z_n(0)) = (0, 0, c_n)$ and $(X_n(1), z_n(1)) = (0, 0, b_n)$. We want to apply Theorem 6 to the sequence $(X_n, z_n)|_{\partial\Delta^+}$, but we can not do this directly.

Proposition 4. *Let (X_n, z_n) be as above then , if k is big enough, a subsequence $(X_{n'})$ converges uniformly to an harmonic map $X : \Delta^+ \to \mathcal{D}$ and $(z_{n'})$ converges uniformly on each compact subset to an harmonic function $z : \overset{\circ}{\Delta^+} \to \mathbb{R}$ (where $\overset{\circ}{\Delta^+}$ is the interior of Δ^+). On $\overset{\circ}{\Delta^+}$, X and z satisfy $z = u \circ X$.*

Proof. In fact the function z_n is not the good function to consider. On $U = [0, \rho_0] \times [\alpha_1, \alpha_2]$, the function Ψ_{u_n} is defined with $\Psi_{u_n}(0, 0) = 0$. Ψ_{u_n} corresponds to the harmonic conjugate of z_n. More precisely, if $\zeta_0 \in \overset{\circ}{\Delta^+}$ and z_n^* is the conjugate function to z_n such that $z_n^*(\zeta_0) = \Psi_{u_n}(X_n(\zeta_0))$, we have $z_n^* = \Psi_{u_n} \circ X_n$. *A priori*, the preceding equality is true only in the interior of Δ^+, but, since Ψ_{u_n} is Lipschitz continuous, z_n^* can be extended to the boundary such that the equality is true everywhere.

By a result of J.C.C. Nitsche [Ni], the area of Σ_n is bounded by $Area(U) + \int_{\partial U} |u_n|$. Since u_n converges to u uniformly on each compact subset of $\Omega \backslash \{\mathcal{Q}\}$, the u_n are uniformly bounded functions on U by Proposition 2. This proves that the areas of Σ_n are uniformly bounded by a constant M. Since (X_n, z_n) are conformal:

$$\int_{\Delta^+} |X_{nx}|^2 + |X_{ny}|^2 + |\nabla z_n|^2 \mathrm{d}x\mathrm{d}y = Area(\Sigma_n)^2 < M^2$$

Let $F_n : \Delta^+ \to \mathcal{D} \times \mathbb{R}^3$ be the map defined by $F_n(\zeta) = (X_n(\zeta), z_n^*(\zeta), x, y)$ with $\zeta = x + iy$. We shall apply Theorem 6 with γ_n the restriction of F_n

to the boundary of Δ^+, $f_n = z_n$ and with $\zeta_1 = -1$, $\zeta_2 = 0$ and $\zeta_3 = 1$. We note $G_n = (F_n, z_n)$. Since z_n^* is conjugate to z_n, $|\nabla z_n^*| = |\nabla z_n|$; then

$$\int_{\Delta^+} \left(|G_{nx}|^2 + |G_{ny}|^2\right) \mathrm{d}x\mathrm{d}y \leq 2M^2 + 2Area(\Delta^+)$$

We have the third hypothesis of Theorem 6.

Because of the function x in F_n, $d(F_n(\zeta_i), F_n(\zeta_j)) \geq 1$ for $i \neq j$, this is the first hypothesis of Theorem 6.

To prove that (γ_n, f_n) satisfies the second hypothesis of Theorem 6, we need a lemma.

Lemma 5. *If k is big enough, for big n we have:*

$$1 \leq \inf_{(0,\rho_0]\times[-2kq\pi,-2(k-1)q\pi]} u_n - \sup_{(0,\rho_0]\times[0,2q\pi]} u_n \tag{5}$$

$$1 \leq \inf_{(0,\rho_0]\times[0,2q\pi]} u_n - \sup_{(0,\rho_0]\times[2kq\pi,2(k+1)q\pi]} u_n \tag{6}$$

Proof. There exists a constant $c \in \mathbb{R}$ such that $u \circ f = u + c$, by Proposition 3 $c \neq 0$ and by construction $c < 0$, this is due to the value $+$ and $-\infty$ on $\widetilde{\mathcal{L}}^+$ and $\widetilde{\mathcal{L}}^-$ for the function u_n. This implies that

$$\sup_{(0,\rho_0]\times[2lq\pi,2(l+1)q\pi]} u - \sup_{(0,\rho_0]\times[2mq\pi,2(m+1)q\pi]} u = c(l-m)$$

$$\inf_{(0,\rho_0]\times[2lq\pi,2(l+1)q\pi]} u - \inf_{(0,\rho_0]\times[2mq\pi,2(m+1)q\pi]} u = c(l-m)$$

This implies that for a k big enough:

$$2 \leq \inf_{(0,\rho_0]\times[-2(k+1)q\pi,-2(k-2)q\pi]} u - \sup_{(0,\rho_0]\times[-2q\pi,4q\pi]} u$$

$$2 \leq \inf_{(0,\rho_0]\times[-2q\pi,4q\pi]} u - \sup_{(0,\rho_0]\times[2(k-1)q\pi,2(k+2)q\pi]} u$$

Let us apply Proposition 2 with $\varepsilon = 1/4$, there exists then d such that for every $l \in \mathbb{Z}$ and every $n > l+2$:

$$\sup_{(0,\rho_0]\times[2lq\pi,2(l+1)q\pi]} u_n \leq \sup_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u_n + \frac{1}{4}$$

$$\inf_{(0,\rho_0]\times[2lq\pi,2(l+1)q\pi]} u_n \geq \inf_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u_n - \frac{1}{4}$$

But since $u_n \to u$ uniformly on every compact subset of $\Omega\backslash\{\mathcal{Q}\}$ we have for big n:

$$\sup_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u_n \leq \sup_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u + \frac{1}{4}$$
$$\inf_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u_n \geq \inf_{[d,\rho_0]\times[2(l-1)q\pi,2(l+2)q\pi]} u - \frac{1}{4}$$

Then in using these inequalities with $l = 0$ and $l = -k$, we have for n big enough:

$$\begin{aligned}
\inf_{(0,\rho_0]\times[-2kq\pi,-2(k-1)q\pi]} u_n &\geq \inf_{[d,\rho_0]\times[-2(k+1)q\pi,-2(k-2)q\pi]} u_n - \frac{1}{4} \\
&\geq \inf_{[d,\rho_0]\times[-2(k+1)q\pi,-2(k-2)q\pi]} u - \frac{1}{2} \\
&\geq \sup_{[d,\rho_0]\times[-2q\pi,4q\pi]} u + \frac{3}{2} \\
&\geq \sup_{[d,\rho_0]\times[-2q\pi,4q\pi]} u_n + \frac{5}{4} \\
&\geq \sup_{(0,\rho_0]\times[0,2q\pi]} u_n + 1
\end{aligned}$$

This shows (5). With $l = 0$ and $l = k$, we get (6):

$$\inf_{(0,\rho_0]\times[0,2q\pi]} u_n \geq \sup_{(0,\rho_0]\times[2kq\pi,2(k+1)q\pi]} u_n + 1$$

$\square$

Let $k \in \mathbb{N}$ given by Lemma 5. In $\mathcal{D}$, the curves $\zeta \mapsto X_n(\zeta)$ for $\zeta \in \partial\Delta^+$ have the same image Γ for every n. Γ is a Jordan curve in $\mathcal{D}$ so for every $m > 0$ there exists $\varepsilon > 0$ such that, if $p', p'' \in \Gamma$ and $d(p', p'') < \varepsilon$, one of the components of $\Gamma\backslash\{p', p''\}$ is of diameter less than m. Let δ be $\min\{\rho_0/2, 1/2\}$ if $\mathcal{A}$ is the point of coordinates (ρ_0, α_1) then 2δ is less than the distance between $\mathcal{O}$ and $\mathcal{A}$ and if $m < \delta$, in the above property, there is only one component of $\Gamma\backslash\{p', p''\}$ with diameter less than m.

Let $0 < m < \delta$, there exists $\varepsilon > 0$ that satisfies the above property, there also exists η such that if $\zeta', \zeta'' \in \partial\Delta^+$ and $|\zeta' - \zeta''| < \eta$, one of the components of $\partial\Delta^+\backslash\{\zeta', \zeta''\}$ is of diameter less than m. Since $m < \delta \leq 1/2$ this component is unique. Let $n \in \mathbb{N}$ be big enough such that (5) and (6) are

satisfied. Let ζ' and ζ'' in $\partial\Delta^+$ such that the distance between $G_n(\zeta')$ and $G_n(\zeta'')$ is less than $\min\{\varepsilon, \eta, 1/2\}$. Let us note $F_n(\zeta') = (p', \Psi_{u_n}(p'), x', y')$ and $F_n(\zeta'') = (p'', \Psi_{u_n}(p''), x'', y'')$. We have $d(p', p'') \leq \varepsilon$, then there exists one component of $\Gamma\backslash\{p', p''\}$ with diameter less than m. Let I be the part of $\partial\Delta^+$ that parametrize this component; the end points of I are $\zeta' = x' + iy'$ and $\zeta'' = x'' + iy''$. Since $|\zeta' - \zeta''| < \eta$, I or its complemantary in $\partial\Delta^+$ is of diameter less than m; let us prove that it is I. If $\mathcal{O} \notin X_n(I)$ then $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\} \cap I = \emptyset$, but $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$ is of diameter 2 then I is of diameter less than m. If $\mathcal{O} \in X_n(I)$ then $\mathcal{A} \notin X_n(I)$ and the points p' and p'' are at a distance less than m from $\mathcal{O}$ then the point p' and p'' can not be on the part of $\Gamma : \theta \mapsto (\rho_0, \theta)$. There are different possible cases. First, I can be included in $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$ then since $|\zeta' - \zeta''| < \eta$, I is of diameter less than m. The second case is when $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$ is included in I then we can suppose that $p' \in \{(\rho, \alpha_1)\}_{0 \leq \rho \leq \rho_0}$ and $p'' \in \{(\rho, \alpha_2)\}_{0 \leq \rho \leq \rho_0}$ this implies, by Lemma 5, that $|z_n(\zeta') - z_n(\zeta'')| \geq 2$ which is impossible since the distance between $G_n(\zeta')$ and $G_n(\zeta'')$ is less than $\min\{\varepsilon, \eta, 1/2\} < 2$. For the third case, we can suppose that $\zeta' \in \{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$ and $p'' \in \{(\rho, \alpha_2)\}_{0 \leq \rho \leq \rho_0}$ (all the other cases are given by permutating ζ' and ζ'' and α_1 and α_2). Since $\mathcal{A} \notin X_n(I)$, we have $1 \in I$. Besides z_n is decreasing along $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\}$, then, since $c_n - z_n(\zeta'') \geq 1$, by Lemma 5, and $|z_n(\zeta') - z_n(\zeta'')| \leq 1/2$, $\zeta' \in [0, 1]$. Since $|\zeta' - \zeta''| \leq 1/2$, ζ'' is in the part of $\partial\Delta^+$ with $\Re(\zeta) \geq 0$ then the shortest component of $\partial\Delta^+\backslash\{\zeta', \zeta''\}$ is the one that contains 1 then it is I.

So it was proved that I and $X_n(I)$ are of diameter less than m. Since Ψ_{u_n} is 1-Lipschitz continuous and $z_n^* = \Psi_{u_n} \circ X_n$ the set $z_n^*(I)$ is of diameter less than m. Then, if $\zeta', \zeta'' \in \partial\Delta^+$ are such that the distance between $G_n(\zeta')$ and $G_n(\zeta'')$ is less than $\min\{\varepsilon, \eta, 1/2\}$, one component of $F_n(\partial\Delta^+)\backslash\{F_n(\zeta'), F_n(\zeta'')\}$ is of diameter less than $\sqrt{3}m$; this is the second hypothesis.

Let us apply the result of Theorem 6, the family $\{F_n|_{\partial\Delta^+}\}$ is equicontinuous then there exists a subsequence $(F_{n'}|_{\partial\Delta^+})$ that converges uniformly. Since the F_n are harmonic maps, the sequence $(F_{n'})$ converges uniformly on Δ^+. Then $X_{n'} \to X$ uniformly on Δ^+ and $z_{n'}^* \to z^*$ with X and z^* harmonic map. The fact that z_n and z_n^* are harmonic conjugates implies that $(z_{n'})$ converges to the harmonic conjugate z of z^* and the convergence is uniform on each compact subset of $\overset{\circ}{\Delta}{}^+$. The equality $z = u \circ X$ is just the limit of the equality $z_n = u_n \circ X_n$. This ends the proof of Proposition 4. $\square$

Construction 4. Let us summarize what we have done above. We take $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}^*$. For every $n \in \mathbb{N}$, we consider (X_n, z_n) a conformal

parametrization of the graph of u_n over $[0, \rho_0] \times [\alpha - 2kq\pi, \alpha + 2kq\pi]$ by the upper-half disk Δ^+ such that the part parametrized by $\Delta^+ \cap \mathbb{R}$ is the vertical segment in the boundary which is above $\mathcal{O}$. Then Proposition 4 says us that if we take k big enough we have a subsequence $(X_{n'}, z_{n'})$ that converges to (X, z) as described in Proposition 4. We also have the sequence (z_n^*) and $z_{n'}^* \to z^*$ uniformly on Δ^+. X satisties t$\widetilde{\mathcal{L}}^+$he following property.

Corollary 3. *Let X be constructed as in Construction 4. Then*

$$X(\Delta^+) = [0, r_0] \times [\alpha_1, \alpha_2]$$

Proof. Let $\mathcal{A} \in [0, r_0] \times [\alpha_1, \alpha_2]$ then for every n there exists ζ_n such that $X_n(\zeta_n) = \mathcal{A}$. We have $X_{n'} \to X$ uniformly on Δ^+ and, since Δ^+ is compact, a subsequence n'' of n' is such that $\zeta_{n''} \to \zeta \in \Delta^+$. Then $\mathcal{A} = X_{n''}(\zeta_{n''}) \to X(\zeta)$ and $\mathcal{A} = X(\zeta)$. □

3.2.2 The proof

Before proving Theorem 5, we need a remark

Proposition 5. *Let $\alpha \in \mathbb{R}$ then $u(\rho, \alpha)$ converges as ρ goes to 0.*

Proof. Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ be big enough such that Construction 4 can be done. By construction, we have X_n, X, z_n^* and z^* defined on Δ^+. Let U be a compact neighborhood of 0 in Δ^+ such that $X(U) \subset [0, \rho_0] \times (\alpha - \pi/2, \alpha + \pi/2)$. Using the projection $H : (\rho, \theta) \mapsto (\rho \cos\theta, \rho \sin\theta)$ we can consider X as a harmonic map from Δ^+ to $\mathbb{R}^2$. For n big enough, $X_n(U) \subset [0, r_0] \times (\alpha - \pi/2, \alpha + \pi/2)$, the maps X_n, X, z_n^* and z^* can be extended to U' where U' is the union of U and $\{\zeta \in \mathbb{C} | \bar{\zeta} \in U\}$. X_n and z_n^* are extended by Schwarz reflection principle to harmonic map. Since $X_{n'} \to X$ and $z_{n'}^* \to z^*$ on U we have the same convergence for their extensions to U'. In the same way, the functions z_n and z can be extended to U' and the convergence $z_{n'} \to z$ is uniform on each compact of U'. The map (X, z) on U' gives a minimal surface of $\mathbb{R}^3$. Since the point $(X_n(0), z_n(0))$ is for every $n \in \mathbb{N}$ the limit point of $(\rho, \alpha, u_n(\rho, \alpha))$ as ρ goes to 0. The normal to the minimal surface (X_n, z_n) at the origin is $(\sin\alpha, -\cos\alpha, 0)$. Then, the normal to the minimal surface (X, z) is also $(\sin\alpha, -\cos\alpha, 0)$ at the origin. Then the intersection of the minimal surface $\{X(\zeta), z(\zeta)\}_{\zeta \in U}$ with the vertical plane of equation $x \sin\alpha - y \cos\alpha = 0$ in the neighborhood of the point $(X(0), z(0))$ is composed of a piece of the vertical straight line passing by this point and a curve which is above $\{(\rho, \alpha)\}_{\rho > 0}$ with $(X(0), z(0))$ as end point. Then this curve is, in fact, $\{\rho, \alpha, u(\rho, \alpha)\}_{\rho > 0}$, this implies that $\lim\limits_{\rho \to 0} u(\rho, \alpha) = z(0)$. □

We can then make the proof of Theorem 5

Proof of Theorem 5. . Let $k \in \mathbb{N}$ be big enough such that we can make Construction 4 with $\alpha = 0$. We then get a map $X : \Delta^+ \to \mathcal{N}$ and two functions $z : \overset{\circ}{\Delta^+} \to \mathbb{R}$ and $z^* : \Delta^+ \to \mathbb{R}$. Let I be the part of $\partial\Delta^+$ such that $\zeta \in I \Leftrightarrow X(\zeta) = \mathcal{O}$. Since X is monotone, I is connected and by construction $\{\zeta \in \partial\Delta^+ | \zeta \in \mathbb{R}\} \subset I$. Let ζ_1 and ζ_2 denote the two end-points of I and consider the biholomorphic map $h : \Delta^+ \to \Delta^+$ such that $h(-1) = \zeta_1$, $h(0) = (0)$ and $h(1) = \zeta_2$. If $\widetilde{X} = X \circ h$, $\widetilde{z} = z \circ h$ and $\widetilde{z}^* = z^* \circ h$ (it is obvious that the conjugate harmonic function to $\widetilde{z}$ is $\widetilde{z}^*$), we make only a reparametrization of the minimal surface (X, z). For $\zeta \in \partial\Delta^+$, $\widetilde{X}(\zeta) = \mathcal{O} \Leftrightarrow \zeta \in \mathbb{R}$. Then let consider $\zeta_0 \in \partial\Delta^+ \backslash \mathbb{R}$. We have $\widetilde{X}(\zeta_0) \neq \mathcal{O}$ then since $\widetilde{z}(\zeta) = u \circ \widetilde{X}(\zeta)$ we can define $\widetilde{z}(\zeta_0)$ by making ζ converging to ζ_0. $\widetilde{z}$ can be also defined at -1 and 1: as $\zeta \in \partial\Delta^+ \backslash \mathbb{R}$ goes to -1 or 1, $X(\zeta)$ goes to $\mathcal{O}$ along $\{(\rho, \alpha_1)\}$ or $\{(\rho, \alpha_2)\}$ and then $\widetilde{z}(\zeta)$ goes to $\lim\limits_{\rho \to 0} u(\rho, \alpha_1) = z(-1)$ or $\lim\limits_{\rho \to 0} u(\rho, \alpha_2) = z(1)$. Since $\widetilde{X} = \mathcal{O}$ on $\mathbb{R} \cap \Delta^+$, $\widetilde{z}^* = 0$ on the same set. then we can extend $\widetilde{z}^*$ to the whole disk by Schwartz reflection principle, this prove that $\widetilde{z}$ can be also extended to the interior of the whole disk by reflection. Since we have define $\widetilde{z}$ on the circular part of the boundary of Δ^+, $\widetilde{z}$ is then defined also on the boundary of the disk and is continuous on the boundary then $\widetilde{z}$ is the harmonic extention to the disk of this countinuous function on the circle. This proves that $\widetilde{z}$ is a continuous function on Δ^+. Then we have $(\widetilde{X}, \widetilde{z})_{\Delta^+}$ which is a parametrization of the graph of u above $[0, \rho_0] \times [\alpha_1, \alpha_2]$ (because $\widetilde{z} = u \circ \widetilde{X}$) and $(\widetilde{X}, \widetilde{z})_{\Delta^+}$ has a boundary such that the part of this boundary which is above $\mathcal{O}$ is a vertical segment. This proves Theorem 5 because $[0, \rho_0] \times [\alpha_1, \alpha_2]$ contains several period of the domain $\mathcal{N}$. $\square$

3.3 Property of the solution 3

Let u be a solution of the Dirichlet problem asked in Theorem 3. We then can understand the boundary behaviour of the graph of u in $\mathbb{R}^3$.

From Theorem 4, we know that, over the neighborhood $\mathcal{N}$ of $\mathcal{Q}$, the graph is bounded by the vertical straight-line passing by the point $\psi(Q)$.

The other points where there are boundary components for the graph of u are the vertices of Ω. This points statisfy the hypotheses of Theorem 3 in [Ma1]. Then the graph of u is bounded by vertical straight-lines near the vertices.

The last remark we can make is the following. We have $\Psi_u(\mathcal{Q}) = 0 = \Psi_u(\mathcal{V})$; this implies that the conjugate surface of the graph of u, which

is bounded by the conjugate of the boundary of the graph of u, has its boundary included in the plane $\{z = 0\}$.

This remark implies that, if Σ is the graph of u over Ω_0^1, a period of Ω, its conjugate surface can be extend by symmetry with respect to the plane $\{z = 0\}$, we note Σ^* this symmetric surface. We then have the following result.

Lemma 6. *Σ^* is of finite total curvature and its total curvature is $4\pi r$.*

Proof. Using arguements given in [Ma1], we can prove that Σ is of finite total curvature, this implies that the same is true for Σ^*. Then, as in Proposition 2.2 in [HK], each catenoidal end gives a contribution of 2π to the total curvature (see [JM], for the original arguements) and, using Gauss-Bonnet Theorem, we compute the value of the total curvature and get $4\pi r$. □

4 The period problem

4.1 The general case

We now try to build a r-noid with genus 1 and horizontal ends of type I for a given polygon of flux.

Let $V = (v_1, \dots, v_r)$ be a polygon that bounds a multi-domain with cone singularity (D, Q, ψ). Using Construction 1 and Construction 3 as in Section 2, we get a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$ and a solution u on Ω of the Dirichlet problem asked in Theorem 3; as in Section 2, we assume that the period $2q\pi$ of Ω is the angle at the cone singularity of D. Let us consider Ω_0^1 one period of the multi-domain Ω and Σ the graph of u over Ω_0^1. We know, because of the result of the preceding sections, that Σ is a minimal surface bounded by $r-1$ vertical lines passing by the vertices P_i $(i \neq 1)$ of the polygon V, two vertical half-lines over P_1, a vertical segment over $\psi(Q)$ and two curves over the segment $[\psi(Q), \psi(P_1)]$. The conjugate surface of Σ is included in $\{z \geq 0\}$ and the conjugates of the $r-1$ vertical lines, the two vertical falf-lines and the vertical segment are exactly the intersection of the conjugate surface with the plane $\{z = 0\}$. We then can extend the conjugate surface by symmetry with respect to this plane, we get a new surface that we note Σ^*. Σ^* is a solution for the Plateau problem at infinity for the data V if the two components of boundary of Σ^*, coming, by conjugation, from the two curves which are over $[\psi(Q), \psi(P_1)]$, glue together such a way that Σ^* has no boundary.

In fact this two components of boundary differ from a translation, how can we compute the vector of the translation? From the function u we

can derive three closed 1-forms $\mathrm{d}X_1^*$, $\mathrm{d}X_2^*$ and $\mathrm{d}X_3^*$ on Ω which are the differential of the three coordinate functions of the conjugate surface to the graph of u (These 1-forms depend only on the first derivatives of u). For example, we have $\mathrm{d}X_3^* = \mathrm{d}\Psi_u$. In $\mathcal{N}$, the neighborhood of $\mathcal{Q}$ in Ω, we consider the path $\Gamma : \theta \mapsto (\rho, \theta)$ for some $\rho < \rho_0$ and $\theta \in [0, 2q\pi]$; Γ is a lift of a generator of $\pi_1(D\backslash\{Q\})$. Then the two components of boundary of Σ^* differ from the following vector, called the *period vector*:

$$\left(\int_\Gamma \mathrm{d}X_1^*,\ \int_\Gamma \mathrm{d}X_2^*,\ \int_\Gamma \mathrm{d}X_3^*\right)$$

Since $\mathrm{d}X_3^* = \mathrm{d}\Psi_u$ and as Ψ_u is invariant by f, $\displaystyle\int_\Gamma \mathrm{d}X_3^* = 0$. Obviously, the value of the integrals is the same for every Γ which is the lift of a generator of $\pi_1(D\backslash\{Q\})$; in fact, from Remark 2, since $\mathrm{d}X_i^*$ depends only on the derivatives of u, $\mathrm{d}X_i^*$ is well defined on $D\backslash Q$

Then the question of the existence of a solution to the Plateau problem at infinity for the data V becomes: knowing if there exists (D, Q, ψ) bounded by V such that the associated period vector is zero; this is the period problem.
Remark 3. We now give some explanations on Remark 1 and the hypothesis on D saying that its angle at the cone singularity is the period of Ω. Let $V = (v_1, ..., v_r)$, (D, Q, ψ), $(\Omega, \mathcal{Q}, \varphi)$ and u be as above (D sastisfies the hypothesis H). We also suppose the the period vector associated to D is zero. Then Σ^* which is the conjugate surface to Σ, the graph of u over Ω_0^1, extended by symmetry is a r-noid with genus 1 and horizontal ends of type I having V as flux polygon. Let f be the isometry of Ω associated to its periodicity. Let $a \in \mathbb{N}^*$, then the quotient of $\mathcal{W}$ by the group $\{f^{an}\}_{n\in\mathbb{Z}}$ ($\mathcal{W}$ is given by Construction 1 applied to D) is a multi-domain with cone singularity that bounds the polygon

$$V_a = (\underbrace{v_1, \dots, v_r, \dots\dots, v_1, \dots, v_r}_{a \text{ times}})$$

Besides, if Σ_a is the graph of u on Ω_0^a, the conjugate surface Σ_a^* of Σ_a extended by symmetry is a ar-noid with genus 1 and horizontal ends of type I having V_a as flux polygon. In fact Σ_a^* is just Σ^* that we cover a times. Then we also can find solution to the Plateau problem at infinity for D that does not satisfy the hypothesis H.

4.2 The period map and the proof of Theorem 2

In this subsection we explain how we shall prove Theorem 2.

Let $V = (v_1, \dots, v_r)$ be a polygon bounded by an immersed polygonal disk $(\mathcal{P}, \psi)$. For each A in the interior of $\mathcal{P}$, Construction 1 gives us $(\mathcal{W}_A, \mathcal{A}, \varphi_A)$ a multidomain with a logarithmic singularity . Then by applying Construction 3 and Theorem 3, we get a periodic multi-domain with logarithmic singularity $(\Omega_A, \mathcal{A}, \varphi_A)$ and a function u_A defined on Ω_A. We then have the three closed 1-forms $\mathrm{d}X_i^*(A)$. To prove Theorem 2, we need to find a point $A \in \overset{\circ}{\mathcal{P}}$ such that the period vector associated to the above constuction for A is 0.

In fact, we have a map from the interior $\mathcal{P}$ to $\mathbb{R}^2$ which associates to every point $A \in \overset{\circ}{\mathcal{P}}$ the vector $\left(\int_\Gamma \mathrm{d}X_1^*(A),\ \int_\Gamma \mathrm{d}X_2^*(A)\right)$, this map is the period map and will be noted Per. Then to prove Theorem 2, the problem is to prove that this map vanishes at one point of $\overset{\circ}{\mathcal{P}}$. The period map satisfies the following proposition.

Proposition 6. *The period map Per is continuous on the interrior of $\mathcal{P}$.*

Proof. Let us consider a sequence (A_n) of points in $\overset{\circ}{\mathcal{P}}$ that converges to $A \in \overset{\circ}{\mathcal{P}}$. Let Γ be a closed path in $\mathcal{P} \backslash \{A, A_0, A_1, \dots, A_n, \dots\}$ such that for every n, Γ is a generator of $\pi_1(\mathcal{P}\backslash\{A_n\})$ and Γ is a generator of $\pi_1(\mathcal{P}\backslash\{A\})$. Since for A (or every A_n) we have $u_A \circ f = u_A + c$ the derivatives of u_A are well defined on $\mathcal{P}\backslash\{A\}$. Then the two closed 1-forms $\mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A)$ are well defined on $\mathcal{P}\backslash\{A\}$. The same is true for A_n. We then have:

$$Per(A_n) = \left(\int_\Gamma \mathrm{d}X_1^*(A_n),\ \int_\Gamma \mathrm{d}X_2^*(A_n)\right)$$

On $\mathcal{P}$, we have the sequence of the derivatives of u_{A_n} and these derivatives converge to the derivatives of u_A if there is no line of divergence (a line of divergence is a phenomenon linked to the behaviour of the first derivatives so we can use this arguement in this case). Since the arguements used in the proof of the existence part of Theorem 3 are always true, there is no line of divergence and $\mathrm{d}X_1^*(A_n) \to \mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A_n) \to \mathrm{d}X_2^*(A)$, the convergence is uniform along Γ. This proves that $Per(A_n) \to Per(A)$ by integration. $\square$

The idea to prove that the period map vanishes at one point in the interior of $\mathcal{P}$ is then to extend continuously Per to the boundary of $\mathcal{P}$ and show that the degree of the period map along the boundary of $\mathcal{P}$ is non zero. In fact, we shall use a modified boundary of $\mathcal{P}$. Then, using Proposition 3.20

in [Fu], this proves that there exists a point $A \in \overset{\circ}{\mathcal{P}}$ where $Per(A) = 0$. The following section is devoted to the extension of Per to the boundary and to the proof of Theorem 7 that establishes that the degree of the period map is non-zero along the boundary.

To extend the period map on the boundary we make a renormalization of the map Per: let A be a point of $\overset{\circ}{\mathcal{P}}$, if $||Per(A)|| \leq 1$ then we do not change the value of $Per(A)$ but if $||Per(A)|| \geq 1$ the new value of $Per(A)$ is $\dfrac{Per(A)}{||Per(A)||}$. The new period map is always continuous and for every point the norm of the period at this point is less than one.

5 The period map on the boundary of $\mathcal{P}$

We use the notations introduced in Subsection 4.2.

5.1 The behaviour on the edges

Proposition 7. *Let (A_n) be a sequence in $\overset{\circ}{\mathcal{P}}$ such that $A_n \to A$ where A is a point in the interior of one edge of the boundary of $\mathcal{P}$. Then $Per(A_n)$ converges to $\mathrm{d}\psi|_A(N)$ where N is the outer unit normal to the edge at A, we recall that ψ is the developping map of $\mathcal{P}$.*

Proof. As in [Ma1], we note $\Omega(\mathcal{P})$ the multi-domain obtained when we glue to every edge $[P_i, P_{i+1}]$ a half strip isometric to $[P_i, P_{i+1}] \times \mathbb{R}_+$. Then, for every A_n the covering map $\pi : \mathcal{W}_{A_n} \to \mathcal{P}$ extends to a covering map $\pi : \Omega_{A_n} \to \Omega(\mathcal{P})$ and the derivatives of u_{A_n} are then well defined on $\Omega(\mathcal{P}) \backslash \{A_n\}$, by Remark 2. Suppose that the point A is in the interior of the edge $[P_1, P_2]$ and that $|P_1P_2| = 2$. By choosing a good chart, we can suppose that $[-1,1] \times \mathbb{R}_+$ is the half-strip glued to this edge. We note D_r the domain in $\mathbb{R}^2$ which is the intersection of the domain $y \leq 0$ and the disk of center $(0, r)$ and radius $\sqrt{r^2+1}$. By choosing a $r > 0$ big enough, D_r is a neighborhood of the edge $[P_1, P_2]$ in $\mathcal{P}$.

We suppose that A is the point $(a, 0)$ $(-1 < a < 1)$ and the points A_n lie in D_r and have coordinates (a_n, b_n) $(b_n < 0)$. We have $a_n \to a$ and $b_n \to 0$. For every n, we note L_n the half straight-line $\{(a_n, b_n + t)\}_{t \geq 0}$ and L the half straight-line $\{(a, t)\}_{t \geq 0}$. By using the covering map π and u_{A_n}, we can define on $\Omega(\mathcal{P}) \backslash L_n$ a function u_n which has the same derivatives as u_{A_n}; u_n is solution of the minimal surface equation and has the value $+\infty$ (resp. $-\infty$) on $\pi(\mathcal{L}_i^+)$ (resp. $\pi(\mathcal{L}_i^-)$). We then want to understand the convergence

of u_n. By the same arguement as in the proof of Theorem 3, there is no line of divergence in $\Omega(\mathcal{P})\backslash L$; this proves that, for a subsequence, (u_n) converges to a solution u of (MSE) on $\Omega(\mathcal{P})\backslash L$. By Lemma A.2, u takes the value $+\infty$ (resp. $-\infty$) on $\pi(\mathcal{L}_i^+)$ (resp. $\pi(\mathcal{L}_i^-)$). We then need to understand its behaviour near L to know the function u.

Since by convention $\Psi_{u_n}(A_n)=0$, we fix $\Psi_u(A)=0$

Lemma 7. *With this convention, $\Psi_u(a,t)=t$ for $t\geq 0$.*

Proof. Since Ψ_u is 1-Lipschitz continuous, $\Psi_u(a,t)\leq t$; let us suppose that for some t_0 we have $\Psi_u(a,t_0)=t_0-\varepsilon$ with $\varepsilon>0$, then for $t>t_0$ $\Psi_u(a,t)\leq t-\varepsilon$. We have $\Psi_u(1,t)=t$ for every $t\geq 0$ because u takes the value $+\infty$ along $\pi(\mathcal{L}_0^+)$. We have:

$$\Psi_u(1,t)-\Psi_u(a,t)=\lim_{n\to+\infty}\Psi_{u_n}(1,t)-\Psi_{u_n}(a,t)=\lim_{n\to+\infty}\int_{[(a,t),(1,t)]}\mathrm{d}\Psi_{u_n}$$

By Lemma 2, the integral is always less than $2\sqrt{2}\frac{1-a}{t}$ for big t. So for t big enough, this upper-bound is less than ε: this give us a contradiction. $\square$

The result of Lemma 7 says us, by Lemma A.2, that u takes the value $+\infty$ on one side of L and $-\infty$ on the other side; more pricisely, $u(a+\eta,t)$ tends to $+\infty$ (resp. $-\infty$) if η tends to 0 by negative value (resp. positive value). There is only one solution for the Dirichlet problem for such boundary condition (we apply Theorem 7 in [Ma1] with the polygon $(\overrightarrow{P_1A},\overrightarrow{AP_2},v_2,\cdots,v_r)$), this proves that the limit for the subsequences of (u_n) is unique then the sequence (u_n) must converge to the function u.

In $[-1,1]\times\mathbb{R}_+\cup D_r$, the 1-form $\mathrm{d}X_1^*(A_n)$ and $\mathrm{d}X_2^*(A_n)$ are given by:

$$\mathrm{d}X_1^*(A_n)=\frac{q_np_n}{W_n}\mathrm{d}x+\frac{1+q_n^2}{W_n}\mathrm{d}y$$
$$\mathrm{d}X_2^*(A_n)=-\frac{1+p_n^2}{W_n}\mathrm{d}x-\frac{p_nq_n}{W_n}\mathrm{d}y$$

with p_n and q_n the first derivatives of u_n (see [Os]). Using this expressions, we can also define the 1-forms $\mathrm{d}X_1^*(A)$ and $\mathrm{d}X_2^*(A)$.

Let η_1 be a small positive number, $\eta_2<\eta_1$ and l positive numbers. Let Γ be the closed path which consists in the segment $[(a+\eta_1,l),(a-\eta_1,l)]$, the segment $[(a-\eta_1,l),(a-\eta_1,0)]$, the half circle in D_r of center A and radius η_1 and the segment $[(a+\eta_1,0),(a+\eta_1,l)]$. Besides we call Γ_1 the part of Γ consisting in the two vertical segments and the half circle, Γ_2 the union of the two segments $[(a+\eta_1,l),(a+\eta_2,l)]$ and $[(a-\eta_2,l),(a-\eta_1,l)]$ and

Γ_3 the segment $[(a+\eta_2, l), (a-\eta_2, l)]$ (see Figure 2). There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and every η_1, η_2 and l, the period for the point A_n is computed by

$$\left(\int_\Gamma \mathrm{d}X_1^*(A_n), \int_\Gamma \mathrm{d}X_2^*(A_n)\right)$$

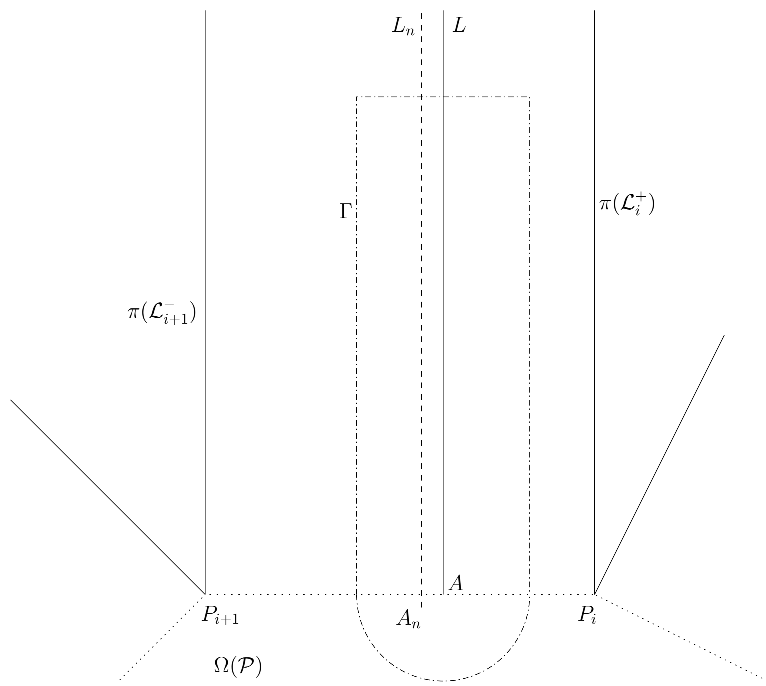

Figure 2:

Let $0 < \alpha < 1$. By Lemma 2, we have $\dfrac{|q_n|}{|p_n|} \leq \sqrt{2}\dfrac{2}{l}\dfrac{1}{1-\frac{4}{l^2}}$ on $\Gamma_2 \cup \Gamma_3$. Then in choosing l big enough, we can ensure that:

$$\left|\int_{\Gamma_2+\Gamma_3} \mathrm{d}X_1^*(A_n)\right| < \frac{\alpha}{2}\int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)$$

We have:

$$\lim_{n\to+\infty}\int_{\Gamma_1} \mathrm{d}X_1^*(A_n) = \int_{\Gamma_1} \mathrm{d}X_1^*(A)$$
$$\lim_{n\to+\infty}\int_{\Gamma_1} \mathrm{d}X_2^*(A_n) = \int_{\Gamma_1} \mathrm{d}X_2^*(A)$$

the same is true on Γ_2 and as η_2 tends to 0, we have $\displaystyle\int_{\Gamma_2} \mathrm{d}X_2^*(A) \to +\infty$. The last assertion is due to Lemma 1 in [JS] which implies that, as η goes to zero $\dfrac{p}{W}(a+\eta,l) \longrightarrow 1$ and $p(a+\eta,l) \geq \dfrac{C}{\eta}$ for some constant C. We then can choose η_2 such that for big n, we have

$$\frac{\left|\int_{\Gamma_1} \mathrm{d}X_1^*(A_n)\right|}{\int_{\Gamma_2} \mathrm{d}X_2^*(A_n)} < \frac{\alpha}{2} \quad \text{and} \quad \frac{\left|\int_{\Gamma_1} \mathrm{d}X_2^*(A_n)\right|}{\int_{\Gamma_2} \mathrm{d}X_2^*(A_n)} < \frac{\alpha}{8}$$

This implies first that $\displaystyle\lim_{n\to+\infty}\int_{\Gamma} \mathrm{d}X_2^*(A_n) = +\infty$, since $\displaystyle\int_{\Gamma_3} \mathrm{d}X_2^*(A_n) \geq 0$; then the period at A_n, for big n, is renormalized and must have a non negative second coordinate. Secondly, for big n, we have:

$$\begin{aligned}
\frac{\left|\int_{\Gamma} \mathrm{d}X_1^*(A_n)\right|}{\left|\int_{\Gamma} \mathrm{d}X_2^*(A_n)\right|} &\leq \frac{\left|\int_{\Gamma_1} \mathrm{d}X_1^*(A_n)\right| + \left|\int_{\Gamma_2} \mathrm{d}X_1^*(A_n)\right| + \left|\int_{\Gamma_3} \mathrm{d}X_1^*(A_n)\right|}{-\left|\int_{\Gamma_1} \mathrm{d}X_2^*(A_n)\right| + \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \int_{\Gamma_3} \mathrm{d}X_2^*(A_n)} \\
&\leq \frac{\frac{\alpha}{2}\int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \frac{\alpha}{2}\int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \alpha\int_{\Gamma_3} \mathrm{d}X_2^*(A_n)}{-\frac{\alpha}{8}\int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + \int_{\Gamma_2} \mathrm{d}X_2^*(A_n) + (1-\frac{\alpha}{8})\int_{\Gamma_3} \mathrm{d}X_2^*(A_n)} \\
&\leq \frac{\alpha\int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)}{(1-\frac{\alpha}{8})\int_{\Gamma_2+\Gamma_3} \mathrm{d}X_2^*(A_n)} \\
&\leq \frac{\alpha}{1-\frac{\alpha}{8}}
\end{aligned}$$

This proves that the renormalized period converges to the vector $(0,1)$; this is what we want to prove. □

5.2 The behaviour at the vertices

Because of Proposition 7, it is clear that we can not extend the period map to the vertices and obtain a continuous map on the boundary. In fact the idea to solve this problem is to make a blowing-up of $\mathcal{P}$ at its vertices.

Let P_i be a vertex of $\mathcal{P}$, there exists $\alpha > 0$ such that a neighborhood of P_i is isometric to $\{(\rho, \theta),\ 0 \leq \rho < \mu, 0 \leq \theta \leq \alpha\}$ with the polar metric $\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2$. A blowing-up at P_i consists in remplacing the point P_i with the segment of all the points $(P_i, \theta)_{0 \leq \theta \leq \alpha}$ and if (A_n) is a sequence of points of $\overset{\circ}{\mathcal{P}}$ converging to P_i in the original topology we shall say that (A_n) converges to (P_i, θ) if $\theta_n \to \theta$ where (ρ_n, θ_n) are the coordinates of A_n near P_i. If we make this blowing-up at all the vertices, we get a new topological space that we note $\widetilde{\mathcal{P}}$; $\widetilde{\mathcal{P}}$ is always a topological space homeomorphic to the closed unit disk and its interior is equal to the interior of $\mathcal{P}$. Then the question is to understand what is the limit of $Per(A_n)$ when A_n tend to some (P_i, θ).

Let us consider the case where $i = 1$; a neighborhood of P_1 in $\mathcal{P}$ is $\{(\rho, \theta),\ 0 \leq \rho < \mu, 0 \leq \theta \leq \alpha\}$. We know that there is a bijection between the Alexandrov-embedded r-noid with genus 0 and horizontal ends and the polygonal immersed disk (see [CR]). In this bijection, the corresponding r-noid to $\mathcal{P}$ will be noted $\Sigma(\mathcal{P})$ and $\Sigma(\mathcal{P})^+$ is the conjugate of a graph over the multi-domain $\Omega(\mathcal{P})$ which contains $\mathcal{P}$ and has as vertices the vertices of $\mathcal{P}$. Besides this graph is bounded by r vertical straight-lines passing by the vertices of $\mathcal{P}$ (see [CR] and [Ma1]). Let us consider $\mathcal{C}$ the conjugate of the straight-line passing by P_1. $\mathcal{C}$ is a strictly convex curve and there exists $\gamma : (-\pi/2, \alpha + \pi/2) \to \{z = 0\}$ a parametrization of $\mathcal{C}$ by its normal. We then have the following result.

Proposition 8. *Let (A_n) be a sequence of points in the interior of $\mathcal{P}$ converging to the point (P_1, θ) of $\partial\widetilde{\mathcal{P}}$. Then $Per(A_n)$ converges to:*

- $(0, -1)$ *if* $\theta = 0$,
- $(-\sin\alpha, \cos\alpha)$ *if* $\theta = \alpha$,
- $\overrightarrow{\gamma(\theta - \pi/2)\gamma(\theta + \pi/2)}$ *or* $\dfrac{\overrightarrow{\gamma(\theta - \pi/2)\gamma(\theta + \pi/2)}}{||\overrightarrow{\gamma(\theta - \pi/2)\gamma(\theta + \pi/2)}||}$, *following the sign of* $||\overrightarrow{\gamma(\theta - \pi/2)\gamma(\theta + \pi/2)}|| - 1$, *if* $\theta \in (0, \alpha)$.

5.2.1 Preliminaries

Let (A_n) be a sequence of points in $\overset{\circ}{\mathcal{P}}$ converging to (P_1, θ) and we suppose that a neighborhood of P_1 in $\mathcal{P}$ is $\{(\rho, \theta),\ 0 \leq \rho < \mu, 0 \leq \theta \leq \alpha\}$. In $\Omega(\mathcal{P})$, a neighborhood of P_1 is then $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \mu) = \{(\rho, \theta),\ 0 \leq \rho < \mu, -\frac{\pi}{2} \leq \theta \leq \alpha + \frac{\pi}{2}\}$. Let u_n be the restriction of the solution u_{A_n} to the period Ω_{A0}^1; we can remark that the period Ω_{A0}^1 can be identified with $\Omega(\mathcal{P}) \backslash [A_n, P_1]$ in using the covering map π.

We note Σ_n the graph in $\mathbb{R}^3$ of u_n and Σ_n^* the minimal surface consisting in the union of the conjugate surface of Σ_n with its symetric with respect to $\{z=0\}$ (the conjugate surface to Σ_n is normalized such that the conjugates of the vertical lines satisfy $z=0$). We also note $\widetilde{\Sigma}_n^*$ the periodic minimal surface consisting in the union of the conjugate surface of the graph of u_{A_n} with its symetric with respect to $\{z=0\}$. In a certain way, Σ_n^* is a period of $\widetilde{\Sigma}_n^*$, and the vector that lets $\widetilde{\Sigma}_n^*$ invariant is the non renormalized $Per(A_n)$.

From Lemma 6, we remark that the total curvature of Σ_n^* does not depend on n and is $4\pi r$.

5.2.2 The convergence of the graphs

To understand the behaviour of the surface Σ_n^* when n goes to $+\infty$ we need to know the behaviour of the sequence (u_n). We consider u_n as a function on $\Omega(\mathcal{P})\backslash[A_n,P_1]$; then the study of the convergence is on the limit multi-domain $\Omega(\mathcal{P})$. Using the arguements of the proof of Theorem 3, we see that there is no line of divergence so a subsequence $(u_{n'})$ converges to a function u solution of (MSE) on $\Omega(\mathcal{P})$ and taking the value $+\infty$ on $\pi(\mathcal{L}_i^+)$ and $-\infty$ on $\pi(\mathcal{L}_i^-)$ (more precisely, if B is a point in $\mathcal{P}$ we have $u_{n'}-u_{n'}(B)\rightarrow u$). By Theorem 7 in [Ma1], such a solution u is unique; then, in fact, the sequence (u_n) converges to u. The graph of this function is the conjugate surface to $\Sigma(\mathcal{P})^+$.

We also need to study the convergence of (u_n) near the point P_1, and to do this we shall renormalize a neighborhood of P_1. u_n is defined on the neighborhood $T(-\frac{\pi}{2},\alpha+\frac{\pi}{2},\mu)\backslash[A_n,P_1]$ of P_1; more precisely, the derivatives of u_n are well defined on $T(-\frac{\pi}{2},\alpha+\frac{\pi}{2},\mu)\backslash\{A_n\}$. We have $A_n=(\rho_n,\theta_n)$ with $\theta_n\rightarrow\theta$ and $\rho_n\rightarrow 0$. We then renormalized by $\frac{1}{\rho_n}$: we get a function v_n defined on $T(-\frac{\pi}{2},\alpha+\frac{\pi}{2},\frac{\mu}{\rho_n})\backslash\{(\rho,\theta_n),\rho\in[0,1]\}$ by $v_n(\rho,\beta)=\frac{1}{\rho_n}u_n(\rho_n\rho,\beta)$. v_n is a solution of the minimal surfaces equation. We want to understand the asymptotic behaviour of v_n. Since $\frac{\mu}{\rho_n}\rightarrow+\infty$ the limit multi-domain is $T(-\frac{\pi}{2},\alpha+\frac{\pi}{2},+\infty)\backslash\{(1,\theta)\}$ for the derivatives of v_n. In the following we note $B(\beta)$ the point of polar coordinates $(1,\beta)$ and $L(\beta)$ the half straight-line $\{(\rho,\beta)\}_{\rho>0}$.

First we must study the lines of divergence. We know that v_n takes the value $+\infty$ on $L(-\frac{\pi}{2})$ and the value $-\infty$ on $L(\alpha+\frac{\pi}{2})$; we have $\Psi_{v_n}(P_1)=0$, $\Psi_{v_n}(B(\theta_n))=0$ and $\Psi_{v_n}\geq 0$. Let L be a line of divergence, L must have an end-point, otherwise we can apply the arguement of the proof of Theorem 3 with the point P_1. This end point can not be on $L(\alpha+\frac{\pi}{2})$ or $L(-\frac{\pi}{2})$ because of Lemma A.1. Then the end point must be P_1 or $B(\theta)$. If the line

of divergence has two end-points, it is the segment $[P_0, B(\theta)]$ then we have

$$0 = |\Psi_{v_n}(B(\theta_n)) - \Psi_{v_n}(P_1)| = \left| \int_{[P_1, B(\theta_n)]} \mathrm{d}\Psi_{v_n} \right| \longrightarrow \left| \int_{[P_1, B(\theta)]} \lim \mathrm{d}\Psi_{v_n} \right| \longrightarrow 1$$

This is a contradiction.

We then can ensure that L is a half straight-line with P_1 or $B(\theta)$ as end-point. Suppose that the end-point is P_1 then L is some $L(\beta)$.

Lemma 8. *Let L be a line of divergence with P_1 as end-point, L is some $L(\beta)$. Then $\beta \notin (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$*

Proof. Let us suppose that $\beta \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$. We note C_ρ the point of $L(\beta)$ with coordinates (ρ, β). Since $\mathrm{d}\Psi_{v_n}$ is closed,

$$\int_{[P_1, B(\theta_n)]} \mathrm{d}\Psi_{v_n} + \int_{[B(\theta_n), C(\rho)]} \mathrm{d}\Psi_{v_n} + \int_{[C(\rho), P_1]} \mathrm{d}\Psi_{v_n} = 0$$

The first integral is always zero, then

$$\left| \int_{[C(\rho), P_1]} \mathrm{d}\Psi_{v_n} \right| \leq |B(\theta_n) C(\rho)|$$

Then by passing to the limit for a subsequence making L appears we get, $\rho \leq |B(\theta)C(\rho)|$, but $|B(\theta)C(\rho)| = \sqrt{\rho^2 + 1 - 2\rho\cos(\beta - \theta)}$ then for big ρ the inequality is not true. $\square$

We suppose now that the end-point of L is $B(\theta)$. We note (ρ', γ') the polar coordinates on $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty)$ with $B(\theta)$ as origin point; γ' is chosen such that the coordinates of P_0 in this new coordinates are $(1, \pi)$. In this polar coordinates, the line of divergence L is some $L'(\beta) = \{\gamma' = \beta,\ \rho' > 0\}$.

Lemma 9. *Let L be a line of divergence with $B(\theta)$ as end point, L is some $L'(\beta)$. Then $\beta \notin (-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.*

Proof. The proof is the same as the one of Lemma 8 in exchanging P_1 and $B(\theta)$. $\square$

In the following, we shall prove that, in fact, all the lines of divergence that we have not excluded by Lemma 8 and 9 yet appear. We first observe that the allowed lines of divergence do not intersect themselves. Since, for every n, $\Psi_{v_n}(P_1) = 0 = \Psi_{v_n}(B(\theta_n))$ and $\Psi_{v_n} \geq 0$, there is only one

possibility for the limit normal on each line of divergence. Besides, we know that $\mathcal{B}(v_n) = \{P \in T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty) | \; |\nabla v_n(P)| \text{ is bounded}\}$ contains a strip which is delimited by the two half straight-lines with P_1 as end-point $L(\theta - \frac{\pi}{2})$ and $L(\theta + \frac{\pi}{2})$ and the two half straight-lines with $B(\theta)$ as end-point $L'(-\frac{\pi}{2})$ and $L'(\frac{\pi}{2})$ for the polar coordinates centred on $B(\theta)$. If all the lines of divergence appear, $\mathcal{B}(v_n)$ is exactly this strip (see Figure 3).

Let us suppose that one allowed line of divergence L do not appear, *i.e.* $L \subset \mathcal{B}(v_n)$. Then the connected component $\mathcal{B}$ of $\mathcal{B}(v_n)$ that contains L is then a multi-domain which is such that there exists a subset K such that $\mathcal{B} \backslash K$ is isometric to an angular sector (let us observe that the angle at the vertex can be greater than 2π) minus the set of the points at a distance less than d from the vertex of the angular sector (d is a positive number). On $\mathcal{B}$ we have a subsequence $v_{n'}$ that converges to some function v. Since $\mathcal{B}$ is bounded by lines of divergence or by the boundary of $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, +\infty)$ the value of v is $+\infty$ on one side of the angular sector and $-\infty$ on the other side, by Lemma A.2. Besides $\Psi_v \geq 0$ since $\Psi_{v_n} \geq 0$ for every n. Then, the function v satisfies many conditions that contradict Theorem 2 in [Ma2]; this proves there is no sub-sequence such that one of the allowed lines of divergence does not appear. We then know the limit of the normal to the graph of v_n for all the points outside the strip.

On the strip, there is a sub-sequence $(v_{n'})$ that converges to some function v (in fact $v_{n'}$ is not well defined on the strip since it is defined only outside $[P_1, B(\theta_n)]$, but the derivatives are well defined and converge to the derivatives of some function v). The function v takes the value $-\infty$ on $L(\theta + \frac{\pi}{2})$ and $L'(-\frac{\pi}{2})$ and the value $+\infty$ on $L(\theta - \frac{\pi}{2})$ and $L'(\frac{\pi}{2})$ by Lemma A.2; such a solution v is unique and is a peace of helicoid. More precisely, if the strip is isometricaly parametrized by $\mathbb{R} \times [-1/2, 1/2]$ with $P_1 = (0, 1/2)$ and $B(\theta) = (0, -1/2)$ then we have $v(x, y) = x \tan(\pi y)$.

Remark 4. Let P be a point in the strip, the curvature of the helicoid at this point is non zero, then the curvature of the graph of v_n over the point P goes to the curvature of the helicoid at this point. This implies that there exists a sequence of point $p_n \in \Sigma_n$ such that the curvature of Σ_n at p_n goes to $+\infty$.

Remark 5. Besides, we know that there exists, for each n, a constant c_n such that $u_{A_n} \circ f_n = u_{A_n} + c_n$, then the result above proves that $\dfrac{c_n}{\rho_n} \to +\infty$.

We also want to know what occurs when we homothetically expand the sequence u_n in a general way. Let (M_n) be a sequence of point in $\Omega(\mathcal{P})$ such that, for every n, $M_n \neq A_n$ and (λ_n) a sequence of positive number such that λ_n goes to $+\infty$. Let h_n be the homothety of center M_n and ratio λ_n, by

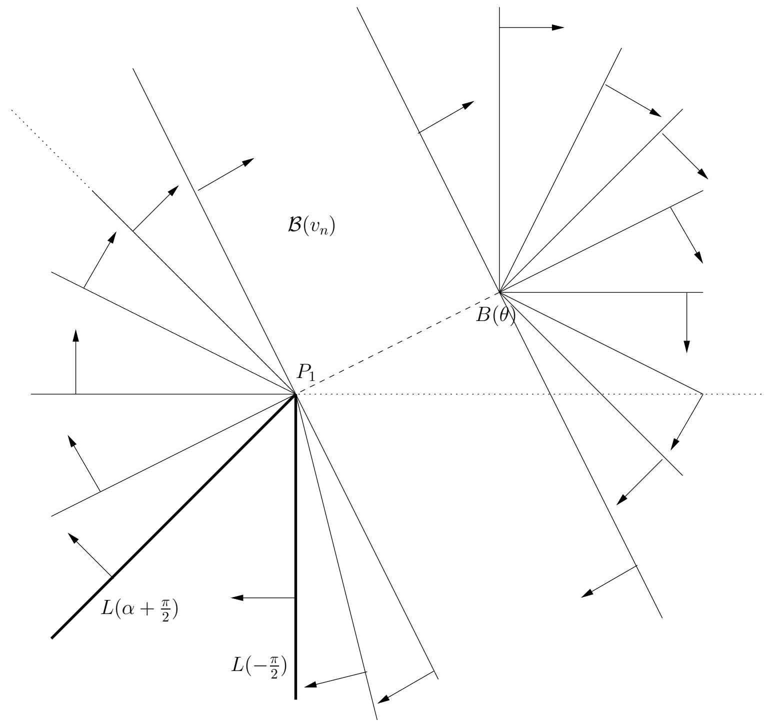

Figure 3: the asymptotic behaviour of v_n

applying the dilatation h_n we get a function v_n on $h_n(\Omega(\mathcal{P}))\backslash\{A_n\}$ defined by $v_n(M) = \lambda_n u_n(h_n^{-1}(M))$. The question we ask is: what asymptotic behaviours are possible? By taking a subsequence, if it is necessary, we can suppose that we are in one of the following cases.

Case 1. For every i, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to +\infty$. In this case the limit multi-domain for the sequence v_n is $\mathbb{R}^2$ then if there is no line of divergence a sub-sequence $(v_{n'})$ must converge to a linear function by Bernstein Theorem. If there is a line of divergence L, each connected component of the domain of convergence of a subsequence $(v_{n'})$is a strip or a half-plane and $(v_{n'})$ converges to a function v with the value $+\infty$ on one side and $-\infty$ on the other side; but such solution of (MSE) does not exist by Proposition 1 in [Ma2] so the domain of convergence is empty and we have only lines of divergence which are all parallel to L and the limit normal is constant on $\mathbb{R}^2$.

Case 2. For every i, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to d \geq$

0. In this case the limit multi-domain is $\mathbb{R}^2$ minus the limit point A' of $h_n(A_n)$. Since $d(M_n, h_n(P_1)) \to +\infty$, we have $d(h_n(A_n), h_n(P_1)) \to +\infty$ then $\lambda_n \rho_n \to +\infty$; by Remark 5, this implies that $\lambda_n c_n$, which is the vertical jump over $[h_n(A_n), h_n(P_1)]$ for the function v_n, goes to $+\infty$. Then the derivatives of v_n can not converge on $\mathbb{R}^2 \backslash \{A'\}$ and there are lines of divergence. Since $\Psi_{v_n}(h_n(A_n)) = 0$ and $\Psi_{v_n} \geq 0$, using arguements that we have already seen, we can ensure that a line of divergence must be a half straight-line with A' as end-point and that there is only one possibility for the limit normal along the line of divergence. If the domain of convergence $\mathcal{B}(v_n)$ is non empty, each connected component of it is an angular sector of $\mathbb{R}^2$; then on one component, a subsequence $(v_{n'})$ converges to a solution v of (MSE) with the value $+\infty$ on one side and $-\infty$ on the other side. By Proposition 2 in [Ma2], such a solution does not exist and we have only the lines of divergence as asymptotic behaviour.

Case 3. There exists $j \neq 1$ such that $d(M_n, h_n(P_j)) \to c \geq 0$. This implies that, for $i \neq j$, $d(M_n, h_n(P_i)) \to +\infty$ and $d(M_n, h_n(A_n)) \to +\infty$; then the limit multi-domain is an angular sector isometric to some $T(0, \beta, +\infty)$ with $P'_j = \lim h_n(P_j)$ as vertex. As above, the lines of divergence must be half straight-lines with P'_j as end-point and the limit normal on a line of divergence is given by the condition $\Psi_{v_n} \geq 0$. As in Case 2, $\mathcal{B}(v_n)$ is empty and the asymptotic behaviour is given by the lines of divergence.

Case 4. $d(M_n, h_n(P_1)) \to c \geq 0$ and $d(h_n(P_1), h_n(A_n)) \to 0$ or $+\infty$. In this case, the limit multi-domain is an angular sector with $P'_1 = \lim h_n(P_1)$ as vertex and the asymptotic behaviour is the same as in Case 3.

Case 5. $d(M_n, h_n(P_1)) \to c \geq 0$ and $d(h_n(P_1), h_n(A_n)) \to c > 0$. The limit-multi-domain is then an angular sector, with P'_1 as vertex, minus the point $A' = \lim h_n(A_n)$. We are in the situation studied above and we know that the asymptotic behaviour is a domain of convergence which is a strip where (v_n) converges to a piece of helicoid and, outside the strip, lines of divergence with A' or P'_1 as end-point (see Figure 3).

5.2.3 The convergence of Σ_n^*

We first fix the notations. Let Q be a point in the interior of $\mathcal{P}$ and q_n the corresponding point in Σ_n. For every n, the value $\Psi_{u_n}(Q)$ is well defined and we normalize Σ_n^* such that q_n^*, the corresponding point to q_n, has coordinates $(0, 0, \Psi_{u_n}(Q))$; Σ_n^* is then still symetric with respect to the plane $\{z = 0\}$. We want to determine the limit of the sequence of minimal surfaces (Σ_n^*).

Let M be a surface in $\mathbb{R}^3$. In the following, when we shall talk about a geodesical disk $D(m, \mu)$ of center $m \in M$ and radius μ, we shall consider

the disk of radius μ in the tangent plane to M at m with the exponential map $\exp_m : T_mM \to \mathbb{R}^3$. Besides, we shall sometimes identify a point in the tangent plane with its image by $\exp_m$.

Let m_n^* be a sequence of point of $\mathbb{R}^3$ such that m_n^* lies in the part of Σ_n^* which is the conjugate of Σ_n (the third coordinate of m_n^* is non-negative). We make the following assumption: $K_{\widetilde{\Sigma}_n^*}(m_n^*) \to +\infty$. Because of Remark 4, we know that such a sequence exists. Let μ be a positive number, and, for each n, we consider $D(m_n^*,\mu)$ the closed geodesical disk of center m_n^* and radius μ in $\widetilde{\Sigma}_n^*$. On $D(m_n^*,\mu)$, the function $a \mapsto |K_{\widetilde{\Sigma}_n^*}(a)|(\mu - d(a,m_n^*))^2$ admits a maximum in the interior of the disk (the distance d is the distance in the tangent space) and let p_n^* be a point where the maximum is reached. Let us note $\lambda_n = \sqrt{|K_{\widetilde{\Sigma}_n^*}(p_n^*)|}$ and $\mu_n = (\mu - d(p_n^*,m_n^*))$. The geodesical disk $D(p_n^*,\mu_n)$ is included in $D(m_n^*,\mu)$ (in fact the image by the exponential map in $\widetilde{\Sigma}_n^*$ of the geodesical disk $D(p_n^*,\mu_n)$ is included in the image of $D(m_n^*,\mu)$); besides since p_n^* realizes the maximum we have $\lambda_n^2\mu_n^2 \geq K_{\widetilde{\Sigma}_n^*}(m_n^*)\mu^2$, then $\lambda_n\mu_n \to +\infty$ and $\lambda_n \to +\infty$. By translating p_n^* to the origin and homothetically expanding the disk $D(p_n^*,\mu_n)$ by the factor λ_n, we obtain a new geodesical disk $D'_n = D(0,\lambda_n\mu_n)$, (D'_n) is a sequence of minimal surfaces. We have $|K_{D'_n}(0)| = 1$. Let R be a positive number and $\tilde{a}$ a point in the geodesical disk $D(0,R)$ of D'_n; we then note a the point in $D(p_n^*,\mu_n)$ corresponding to $\tilde{a}$. $\exp_{p_n^*}(a)$ is included in the image of the disk $D(m_n^*,\mu)$. Then there exists $a' \in D(m_n^*,\mu)$ such that $\exp_{m_n^*}(a') = \exp_{p_n^*}(a)$ and $d(m_n^*,a') \leq d(m_n^*,p_n^*) + d(p_n^*,a)$; we have $K_{\widetilde{\Sigma}_n^*}(a) = K_{\widetilde{\Sigma}_n^*}(a')$. With this notation we then have:

$$\begin{aligned}
|K_{D'_n}(\tilde{a})|(\lambda_n\mu_n - R)^2 \leq |K_{D'_n}(\tilde{a})|(\lambda_n\mu_n - d(\tilde{a},0))^2 &= |K_{\widetilde{\Sigma}_n^*}(a)|(\mu_n - d(a,p_n^*))^2 \\
&\leq |K_{\widetilde{\Sigma}_n^*}(a')|(\mu - d(a',m_n^*))^2 \\
&\leq \lambda_n^2\mu_n^2
\end{aligned}$$

the equality is due to the fact that the function $|K_{\widetilde{\Sigma}_n^*}(\cdot)|(\mu_n - d(\cdot,p_n^*))^2$ is invariant under rescaling. Thus the curvature on D'_n is uniformly bounded on the sequence of geodesical disks $D(0,R)$ of D'_n. Then there exists a subsequence $(D'_{n'})$ that converges to a complete minimal surface that we denote D'_∞; this surface is complete since $\lambda_n\mu_n \to +\infty$. Besides D'_∞ is non flat since at the origin its curvature is -1. Since D'_∞ is non flat there is a point $\tilde{a}$ where the normal has a negative third coordinate; there exists a neighborhood U of $\tilde{a}$ in the tangent plane to D'_∞ at $\tilde{a}$ such that D'_∞ and $D'_{n'}$, for big n', are graphs over U and $D'_{n'} \to D'_\infty$ as graphs. Let $\tilde{a}_{n'}$ be

the sequence of points in $D'_{n'}$ over $\tilde{a}$ as graphs. The normal at $\tilde{a}_{n'}$ to $D'_{n'}$ has a negative third coordinate then before the rescaling $\tilde{a}_{n'}$ correponds to a point $a_{n'}$ which lies in the conjugate of $\Sigma_{n'}$. Let $b_{n'}$ be the point in $\Sigma_{n'}$ corresponding to $a_{n'}$ and $B_{n'}$ the projection on $\Omega(\mathcal{P})$ of $b_{n'}$. The convergence of $D_{n'}$ to D'_∞ near $\tilde{a}$ says us that $\mathcal{B}(v_{n'})$ is non empty where $v_{n'}$ is the rescaled function of $u_{n'}$ with the factor $\lambda_{n'}$ and $B_{n'}$ as origin points. We are then in Case 1 or Case 5 of the preceding subsection, but if it is Case 1 the limit graph would be a plane which is impossible since D'_∞ is non flat. Then it is Case 5 and the limit graph is a piece of helicoid then this implies that D'_∞ is a catenoid. This also implies that $\lambda_{n'} \sim \frac{c}{\rho_{n'}}$ with c some real constant (we recall that $\rho_n = |P_1 A_n|$).
Remark 6. If for example, we take m_n^* such that $K_{\Sigma_n}(m_n^*) = \max K_{\Sigma_n}(\cdot)$ the above arguements show that $\max K_{\Sigma_n}(\cdot) = O(\left(\frac{1}{\rho_n}\right)^2)$.

The boundary of Σ_n^* is composed of two closed paths Γ_n^1 and Γ_n^2, we want to understand the behaviour of this boundary when n goes to $+\infty$. The boundary of the graph Σ_n is composed of $r-1$ straight-lines, they are over the points P_i for $i \neq 1$, and a curve which consists in a half straight-line over P_1 that goes down from the infinity to some point called t_n^1, a curve which is a graph over the segment $[P_1, A_n]$ joining t_n^1 to some point t_n^2, a vertical segment $[t_n^2, t_n^3]$ over A_n, a curve which is a graph over the segment $[P_1, A_n]$ joining t_n^3 to some point t_n^4 (by a vertical translation, this curve is the same as the curve joining t_n^1 to t_n^2) and a half straight-line over P_1 with t_n^4 as end-point and going down to the infinity. Then the path Γ_n^1 (resp. Γ_n^2) consists in the conjugate of the curve joining t_n^1 to t_n^2 (resp. t_n^3 to t_n^4) with its symetric with respect to the plane $\{z = 0\}$.

As in Subsection 5.2.2, we consider v_n the rescaled function with factor $\frac{1}{\rho_n}$ on $T(-\frac{\pi}{2}, \alpha + \frac{\pi}{2}, \frac{r}{\rho_n}) \backslash [P_1, B(\theta_n)]$. By Remark 6, there exists a constant which bounds the curvature on the graph of v_n for every n. Let p_n be a point in the graph of v_n which is above the middle of the segment $[P_1, B(\theta_n)]$ and $\mathcal{Y}_n$ the conjugate of the graph of v_n that we extend by symmetry and periodicity such that the conjugate point of p_n is the origin of $\mathbb{R}^3$. Let R be a positive number and consider the sequence of geodesical disks $D(0, R)$ in $\mathcal{Y}_n$. Since the curvature is uniformly bounded on the disk there exists a subsequence such that the sequence of disks converges, this implies that, using a Cantor diagonal process, there exists a subsequence $(\mathcal{Y}_{n'})$ which converges to some minimal surface $\mathcal{Y}$. Since the graphs of v_n converge near p_n to a piece of an helicoid, $\mathcal{Y}$ is a catenoid whose flux is the vector $2\overrightarrow{P_1 B(\theta)}$. There is only one possible limit for subsequences of $(\mathcal{Y}_n)$ so the sequence $(\mathcal{Y}_n)$

converges to $\mathcal{Y}$. Let us consider the catenoid given in cylindrical coordinates by $(u,v)\mapsto(\frac{1}{\pi}u,v,\frac{1}{\pi}\operatorname{argch}(u))$. Thus $\mathcal{Y}$ is the translated by $(0,0,-\frac{1}{\pi})$ of the image by a rotation of axis $\{z=0,x\cos\theta+y\sin\theta=0\}$ and angle $\frac{\pi}{2}$ of this catenoid. Suppose that p_n is the point in the graph of v_n which is the limit of the points $(1/2,\beta,v_n(1/2,\beta))$ when $\beta\to\theta_n$ with $\beta<\theta_n$. We note $\mathcal{Y}'_n$ the conjugate of the graph of v_n that we extend by symmetry but not by periodicity such that the conjugate of the point p_n is the origin. What we have proved just above implies that $(\mathcal{Y}'_n)$ converges to the part of the catenoid $\mathcal{Y}$ included in $\{x\cos\theta+y\sin\theta\leq 0\}$; this part is noted $\mathcal{Y}_-$. If p_n is build with $\beta>\theta_n$ we get the half catenoid included in $\{x\cos\theta+y\sin\theta\geq 0\}$.

This proves that the rescaled paths $\frac{1}{\rho_n}\Gamma_n^1$ and $\frac{1}{\rho_n}\Gamma_n^2$ converge to two circles of the same radius. Let us call s_n^i the point in Σ_n^* which is the conjugate of t_n^i for $1\leq i\leq 4$. The above proof shows that for every $\varepsilon>0$ and for big n the ball of center s_n^1 (resp. s_n^4) and radius ε contains Γ_n^1 (resp. Γ_n^2) and even a bigger and bigger part of a surface near a half catenoid. We consider the case where p_n is build with $\beta<\theta_n$. If $B(R)$ is the ball of center the origin and radius R, we have $B(R)\cap\mathcal{Y}'_n$ is near from $B(R)\cap\mathcal{Y}_-$ for big n. Besides for big n, $\rho_nR<\varepsilon$ then the ball $B(s_n^1,\varepsilon)$ contains the homothetic by ρ_n of a surface near $B(R)\cap\mathcal{Y}_-$. The same is true for p_n build with $\beta>\theta_n$. This implies that the sequence of total curvatures of the part of Σ_n^* included in this ball has an lower limit bigger than 2π, since the total curvature of an half catenoid is 2π. In fact, we have to remember that near the points s_n^1 and s_n^4 the surface Σ_n^* behaves like small half-catenoid for big n.

We know that (Σ_n^*) is a sequence of symetric minimal surface with finite total curvature $4\pi r$. Each surface Σ_n^* has two connected components of boundary Γ_n^1 and Γ_n^2. We know that the diameter of each component goes to zero and if B_n is a sequence of balls of diameter ε centred at s_n^2 then the sequence of total curvatures of $\Sigma_n^*\cap B_n$ has an lower limit bigger than 2π. By taking a subsequence we can suppose that (s_n^2) and (s_n^3) diverge to the infinity or converge to s_∞^2 and s_∞^3. Then by results explained in [CR], there exist a finite number of distinct properly and simply immersed branched minimal surfaces $M_1,\dots,M_k\subset\mathbb{R}^3$ with finite total curvature, a finite subset $X\in\mathbb{R}^3$ contained in $M=M_1\cup\cdots\cup M_k$ and a subsequence of (Σ_n^*), that we still call (Σ_n^*), such that:

1. (Σ_n^*) converges to M (with finite multiplicity) on compact subsets of $\mathbb{R}^3\backslash(X\cup\{s_\infty^2,s_\infty^3\})$ in the C^m-topology for any positive integer m;

2. on each M_i the multiplicity m_i is well defined and is such that

$$m_1 C(M_1) + \cdots + m_k C(M_k) \leq C(\Sigma_n^*);$$

3. X is the singular set of the limit $\Sigma_n^* \to M$. Given a point $p \in X$, the amount of total curvature of the sequence (Σ_n^*) which disappears through the point p is a positive multiple of 4π.

Since at s_∞^2 and s_∞^3, there is 2π of total curvature that disappear, we have $m_1 C(M_1) + \cdots + m_k C(M_k) \leq C(\Sigma_n^*) - 4\pi = 4\pi(r-1)$ even if s_n^2 or s_n^3 diverges. We know that the sequence of functions u_n on $\Omega(\mathcal{P}) \backslash [P_1, A_n]$ converge on $\Omega(\mathcal{P})$ to the solution u of the Dirichlet problem asked in Theorem 7 of [Ma1]. Then q_n converges to q_∞ the point in the graph of u which is above Q. The conjugate surface to the graph of u, after being extended by symetry with respect to the plane $\{z = 0\}$, is the solution $\Sigma(\mathcal{P})$ to the plateau problem at infinity with genus 0 for the data $\mathcal{P}$. Let q_∞^* be the point in $\Sigma(\mathcal{P})$ that correspond to q_∞. We then have $q_n^* \to q_\infty^*$ and, in a neighborhood of q_∞^*, Σ_n^* converges to $\Sigma(\mathcal{P})$, then $\Sigma(\mathcal{P})$ is one M_i. We then can suppose that $M_1 = \Sigma(\mathcal{P})$, since $C(\Sigma(\mathcal{P})) = 4\pi(r-1)$, $m_1 = 1$ and for, $i \neq 1$, M_i is a plane and besides X is empty. In fact, in [CR], C. Cosín and A. Ros proves that, in such a convergence, no plane can appear. Then finally, $\Sigma_n^* \to \Sigma(\mathcal{P})$.

Since the problems of convergence appear only near the points s_∞^2 and s_∞^3, the curves in $\Sigma(\mathcal{P})$ which are the conjugates of the $r-1$ straight-lines that are over the points P_i, for $i \neq 1$, are the respective limits of the curves in Σ_n^* which are the conjugates of the $r-1$ straight-lines in Σ_n that are over the points P_i, for $i \neq 1$.

5.2.4 The convergence of $Per(A_n)$

$Per(A_n)$ corresponds to the vector that defines the translation under which $\widetilde{\Sigma}_n^*$ is invariant. Then $Per(A_n)$ is $\overrightarrow{s_n^2 s_n^3}$ or $\frac{\overrightarrow{s_n^2 s_n^3}}{||\overrightarrow{s_n^2 s_n^3}||}$, following the value of $||\overrightarrow{s_n^2 s_n^3}||$.

As above, we consider $\mathcal{C}$ the curve in $\Sigma(\mathcal{P})$ which is the conjugate of the vertical straight-line which is over P_1 in the graph of u. This curve is strictly convex then we can parametrized $\mathcal{C}$ by its normal. Then there exists a parametrization $\gamma : (-\frac{\pi}{2}, \alpha + \frac{\pi}{2}) \to \{z = 0\}$ of $\mathcal{C}$ such that the normal to $\gamma(\beta)$ is $(\sin\beta, -\cos\beta, 0)$; we cover $\mathcal{C}$ as we cover the straight-line over P_1 when we go down. We know that $\mathcal{C}$, outside s_∞^2 and s_∞^3, is the limit, in the Hausdorff topology, of a part of the conjugate of the boundary of the graph Σ_n of u_n which is included in $\{z = 0\}$. For each n this set is composed of

three strictly convex arc, we note $\mathcal{C}_n^1$, $\mathcal{C}_n^2$ and $\mathcal{C}_n^3$: $\mathcal{C}_n^1$ is the conjugate of the vertical half straight-line that have t_n^1 as end-point ($\mathcal{C}_n^1$ has s_n^1 as end-point), $\mathcal{C}_n^2$ is the conjugate of the vertical segment $[t_n^2, t_n^3]$ ($\mathcal{C}_n^2$ is joining s_n^2 to s_n^3) and $\mathcal{C}_n^3$ is the conjugate of the vertical half straight-line that have t_n^4 as end-point ($\mathcal{C}_n^3$ has s_n^4 as end-point). As $\mathcal{C}$, this three strictly convex arcs can be parametrized by their normal, then there exist $\gamma_n^1 : (-\frac{\pi}{2}, \theta_n] \to \{z = 0\}$, $\gamma_n^2 : [\theta_n - \pi, \theta_n + \pi] \to \{z = 0\}$ and $\gamma_n^3 : [\theta_n, \alpha + \frac{\pi}{2}) \to \{z = 0\}$ such that γ_n^i parametrized $\mathcal{C}_n^i$ and the normal at the point $\gamma_n^i(\beta)$ is $(\sin\beta, -\cos\beta, 0)$. We have $\gamma_n^1(\theta_n) = s_n^1$, $\gamma_n^2(\theta_n - \pi) = s_n^2$, $\gamma_n^2(\theta_n + \pi) = s_n^3$ and $\gamma_n^3(\theta_n) = s_n^4$. We note I_n^i the definition set of γ_n^i. We then have $I_n^i \to I^i$ where $I^1 = (\frac{\pi}{2}, \theta]$, $I^2 = [\theta - \pi, \theta + \pi]$ and $I^3 = [\theta, \alpha + \frac{\pi}{2})$. The question is: is γ_n^i converging on I^i?

Since there is a half catenoid that appears near the points s_n^i for big n, if $\beta \in (\theta - \frac{\pi}{2}, \theta] \subset I^1$ the curvature at $\gamma_n^1(\beta)$ becomes infinite, the same is true for the sequence of point $(\gamma_n^2(\beta))$ for $\beta \in [\theta - \pi, \theta - \frac{\pi}{2}) \cup (\theta + \frac{\pi}{2}, \theta + \pi] \subset I^2$ and the sequence of points $(\gamma_n^3(\beta))$ for $\beta \in [\theta, \theta + \frac{\pi}{2}) \subset I^3$.

The set $\mathcal{C}\backslash\{s_\infty^2, s_\infty^3\}$ is composed of a finite number of convex arcs. Let a be a point in $\mathcal{C}\backslash\{s_\infty^2, s_\infty^3\}$, at this point the convergence of Σ_n^* to $\Sigma(\mathcal{P})$ well behaves so there exists a neighborhood U of a in the tangent plane to $\Sigma(\mathcal{P})$ at a (this plane is vertical) such that, over U, $\Sigma(\mathcal{P})$ and Σ_n^*, for big n, are graphs and, as graphs, Σ_n^* converge to $\Sigma(\mathcal{P})$. Over U, there is a neighborhood of a in $\mathcal{C}\backslash\{s_\infty^2, s_\infty^3\}$ and this neighborhood is the limit of the part of Σ_n^* which is included in $\{z = 0\}$. Since each unit vector is reached a finite number of times on $\mathcal{C}_n^1 \cup \mathcal{C}_n^2 \cup \mathcal{C}_n^3$, by taking a subsequence n', there exist β, ε and $i \in \{1, 2, 3\}$ such that, for every n', the part of $\Sigma_{n'}^* \cap \{z = 0\}$ which is over U contains $\gamma_{n'}^i(\beta - \varepsilon, \beta + \varepsilon)$. Then the convergence as graphs of Σ_n^* implies that on $(\beta - \varepsilon, \beta + \varepsilon)$ the sequence $(\gamma_{n'}^i)$ converges to some map $\widetilde{\gamma}$ that parametrized a neighborhood of a in $\mathcal{C}$ such that the normal to the point $\widetilde{\gamma}(\omega)$ is $(\sin\omega, -\cos\omega, 0)$ (the convergence is in the C^m topology for every m). Since $\gamma_{n'}^i \to \widetilde{\gamma}$, for every $\omega \in (\beta - \varepsilon, \beta + \varepsilon)$ the curvature at the point $\gamma_{n'}^i(\omega)$ remains bounded. Then, if $i = 1$, $(\beta - \varepsilon, \beta + \varepsilon) \cap [\theta - \frac{\pi}{2}, \theta] = \emptyset$, if $i = 2$, $(\beta - \varepsilon, \beta + \varepsilon) \cap \big([\theta - \pi, \theta - \frac{\pi}{2}] \cup [\theta + \frac{\pi}{2}, \theta + \pi]\big) = \emptyset$ and, if $i = 3$, $(\beta - \varepsilon, \beta + \varepsilon) \cap [\theta, \theta + \frac{\pi}{2}] = \emptyset$.

Then by applying the above arguement to a countable number of points and constructing a subsequence by diagonal Cantor process, there exist

1. k_3 open intervals in $I = (-\frac{\pi}{2}, \alpha + \frac{\pi}{2})$: $J_1, \dots, J_{k_1} \subset I^1\backslash[\theta - \frac{\pi}{2}, \theta]$, $J_{k_1+1}, \dots, J_{k_2} \subset I^2\backslash\big([\theta - \pi, \theta - \frac{\pi}{2}] \cup [\theta + \frac{\pi}{2}, \theta + \pi]\big)$ and $J_{k_2+1}, \dots, J_{k_3} \subset I^3\backslash[\theta, \theta + \frac{\pi}{2}]$, these intervals satisfy $J_j \cap J_l = \emptyset$ if $j \neq l$, and

2. a map $\widetilde{\gamma}$ with value in $\{z = 0\}$, defined on the union of the k_3 intervals,

$\widetilde{\gamma}$ parametrizes $\mathcal{C}\backslash\{s^2_\infty, s^3_\infty\}$ by its normal,

3. a subsequence n',

such that, on J_j, the sequence $(\gamma^i_{n'})$ (for the corresponding i) converges to $\widetilde{\gamma}$; besides, the J_j are maximal in the sense that $\widetilde{\gamma}$ restricted to J_j parametrized one of the convex arcs that composed $\mathcal{C}\backslash\{s^2_\infty, s^3_\infty\}$. The curve $\mathcal{C}$ has total curvature $\alpha+\pi$ and since $\widetilde{\gamma}$ parametrizes by the normal $\mathcal{C}\backslash\{s^2_\infty, s^3_\infty\}$, which has the same total curvature, we have

$$\alpha+\pi = \sum_{j=1}^{k_3} l(J_j) \leq l(I) = \alpha + \pi \tag{7}$$

where $l(J_j)$ is the length of the interval J_j.

• We now assume that $\theta \notin \{0, \alpha\}$. Under this hypothesis,the computation (7) implies that there exist points

$$-\frac{\pi}{2} = \beta_0 < \beta_1 < \cdots < \beta_{k_1} = \theta - \frac{\pi}{2} < \beta_{k_1+1} < \cdots$$
$$\cdots < \beta_{k_2} = \theta + \frac{\pi}{2} < \beta_{k_2+1} < \cdots < \beta_{k_3} = \alpha + \frac{\pi}{2}$$

such that, for every j, $J_j = (\beta_{j-1}, \beta_j)$.

Lemma 10. *for $j \notin \{0, k_1, k_2, k_3\}$, the sequence $(\gamma^i_{n'})$ (for the corresponding i) converges in a neighborhood of β_j in the C^1 topology, then we can extend the definition of $\widetilde{\gamma}$ at β_j.*

Proof. We apply Proposition B.3, if there is no C^1 convergence near β_j, since $\Sigma^*_n \to \Sigma(\mathcal{P})$ in the Hausdorff topology, $\Sigma(\mathcal{P})$ contains a segment. But this is not true then we can extend the definition of $\widetilde{\gamma}$ at β_j. □

We then have $\gamma^1_{n'} \to \widetilde{\gamma}$ on $(-\frac{\pi}{2}, \theta - \frac{\pi}{2})$, $\gamma^2_{n'} \to \widetilde{\gamma}$ on $(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ and $\gamma^3_{n'} \to \widetilde{\gamma}$ on $(\theta + \frac{\pi}{2}, \alpha + \frac{\pi}{2})$.

We shall then study the behaviour near $\theta - \frac{\pi}{2} = \theta'$.

Let $\beta < \frac{\pi}{4}$ be a small positive angle and $\varepsilon > 0$. We apply Proposition B.2 to $\gamma^1_{n'}$ on $(\theta' - \beta, \theta' + \beta)$, then the set $\gamma^1_{n'}(\theta' - \beta, \theta' + \beta)$ is included in an angular sector, we note S, of vertex $\gamma^1_{n'}(\theta' - \beta)$ and angle 2β. We have $\gamma^1_{n'}(\theta' + \beta) \in S$. Then, for big n', because of the behaviour of the sequence $(\Sigma^*_{n'})$ near the point $s^2_{n'} = \gamma^2_{n'}(\theta)$, we have the distance between $\gamma^1_{n'}(\theta' + \beta)$ and $s^2_{n'}$ less than ε and the distance between $\gamma^1_{n'}(\theta' + \beta)$ and $\gamma^2_{n'}(\theta' - \beta)$ less than ε. Besides by applying Proposition B.2 to $\gamma^2_{n'}$ on $(\theta' - \beta, \theta' + \beta)$, we have

that $\gamma^2_{n'}(\theta'-\beta,\theta'+\beta)$ is included in an angular sector of vertex $\gamma^2_{n'}(\theta'-\beta)$ and angle 2β. Since at $\gamma^1_{n'}(\theta'-\beta)$ and $\gamma^2_{n'}(\theta'-\beta)$, the two strictly convex curves $\mathcal{C}^1_{n'}$ and $\mathcal{C}^2_{n'}$ have the same normal and their curvature have the same sign, the angular sector of vertex $\gamma^2_{n'}(\theta'-\beta)$ is just the translation of S by the vector $\overrightarrow{\gamma^1_{n'}(\theta'-\beta)\gamma^2_{n'}(\theta'-\beta)}$. This second angular sector is then included in the set S_ε of points of $\mathbb{R}^2$ which are at a distance less than ε from S. Then by passing to the limit $n'\to+\infty$, we get that $\widetilde{\gamma}\left((\theta'-\beta,\theta'+\beta)\backslash\{\theta'\}\right)$ is included in the set of points in $\mathbb{R}^2$ which are at a distance less than ε from an angular sector which have $\widetilde{\gamma}(\theta'-\beta)$ as vertex and 2β as angle. The points $s^2_{n'}$ are also in this set for big n' (see Figure 4).

We can do the same work in starting from the point $\gamma^2_{n'}(\theta'+\beta)$ and in covering the curves $\mathcal{C}^1_{n'}$ and $\mathcal{C}^2_{n'}$ in the opposite sense. We then get that $\widetilde{\gamma}\left((\theta'-\beta,\theta'+\beta)\backslash\{\theta'\}\right)$ and $s^2_{n'}$, for big n', are included in the set of points in $\mathbb{R}^2$ which are at distance less than ε from an angular sector which have $\widetilde{\gamma}(\theta'+\beta)$ as vertex and 2β as angle. Since β is less than $\frac{\pi}{4}$, the intersection of the two sets we have just build is a compact subset of $\mathbb{R}^2$ which contains $s^2_{n'}$, then s^2_∞ exists. Then $\widetilde{\gamma}(t)$ admits two limits: one when $t\to\theta'$, $t<\theta'$ and one when $t\to\theta'$, $t>\theta'$. This two limit points are in $\mathcal{C}$ and $\widetilde{\gamma}$ parametrized $\mathcal{C}$ except for two points where the normal is different. Since the normals at this two limit points are the same, the two limit points are, in fact, a unique point and then the definition of $\widetilde{\gamma}$ extend at θ'. Now, letting β and ε goes to zero, we see that the limit compact is just $\widetilde{\gamma}(\theta')$ then $\widetilde{\gamma}(\theta')=s^2_\infty$.

In the same way we see that s^3_∞ exists and $\widetilde{\gamma}(\theta+\frac{\pi}{2})=s^3_\infty$. In fact this prove that $\widetilde{\gamma}$ parametrizes $\mathcal{C}$ by its normal on I and then $\widetilde{\gamma}=\gamma$. Then the sequence $(Per(A_n))$ has only one possible cluster point then the sequence converges to the limit given in Proposition 8.

- We now study the case $\theta=0$ (the case $\theta=\alpha$ can be done in the same way). In this case $k_1=0$ and using the same arguements as above we can prove that: s^3_∞ exists, $\gamma=\widetilde{\gamma}$ and $\gamma(\frac{\pi}{2})=s^3_\infty$. Because of the asymptotic behaviour of the curve $\mathcal{C}$, we know that there exists $0<\beta<\frac{\pi}{2}$ such that for every $t\in(-\frac{\pi}{2},-\frac{\pi}{2}+\beta)$, $|\gamma(t)s^3_\infty|>2$ and $\frac{\overrightarrow{\gamma(t)s^3_\infty}}{|\gamma(t)s^3_\infty|}$ is at a distance less than ε from the vector $(0,-1)$. Let $0<\beta'<\beta$, we consider the angular sector S of vertex $\gamma(-\frac{\pi}{2}+\beta')$ and angle 2β which contains $\gamma(-\frac{\pi}{2},-\frac{\pi}{2}+\beta')$ and has, as a part of its boundary, the half straight-line tangent to $\mathcal{C}$ at $\gamma(-\frac{\pi}{2}+\beta')$ and with $\gamma(-\frac{\pi}{2}+\beta')$ as end-points; S exists because of Proposition B.2. Then, because of the asymptotic behaviour of $\mathcal{C}$, there exist $0<\beta'<\beta$ and ε' such that, for every point s at a distance less than ε' from S, we have $|ss^3_\infty|>2$ and $\frac{\overrightarrow{ss^3_\infty}}{|ss^3_\infty|}$ is at a distance less than ε from $(0,-1)$.

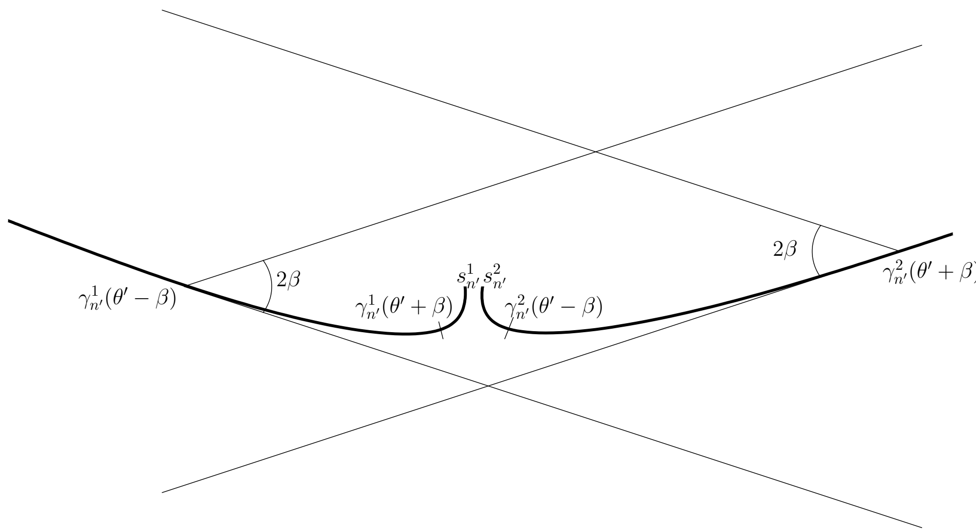

Figure 4: the local behaviour near $s^2_{n'}$

We apply now Proposition B.2 to $\gamma^2_{n'}$ on $(-\frac{\pi}{2}-\beta,-\frac{\pi}{2}+\beta)$ that we cover in the opposite sense. We then obtain that $\gamma^2_{n'}(-\frac{\pi}{2}-\beta,-\frac{\pi}{2}+\beta)$ is included in an angular sector with vertex $\gamma^2_{n'}(-\frac{\pi}{2}+\beta')$ and angle 2β. Since $\gamma^2_{n'}\to\gamma$ on $(-\frac{\pi}{2},-\frac{\pi}{2}+\beta)$, these angular sector converges to S, in fact, the angular sectors we have build are just the translations of S by the vector $\overrightarrow{\gamma(-\frac{\pi}{2}+\beta')\gamma^2_{n'}(-\frac{\pi}{2}+\beta')}$ which goes to zero. Besides we know that the distance between $\gamma^2_{n'}(-\frac{\pi}{2}-\beta')$ and $s^2_{n'}$ goes to zero. Then for big n', $s^2_{n'}$ is at a distance less than ε' from S. Then, using that $s^3_{n'}$ goes to s^3_∞, this proves that for big n', $|s^2_{n'}s^3_{n'}|>2$ and $\frac{\overrightarrow{s^2_{n'}s^3_{n'}}}{|s^2_{n'}s^3_{n'}|}\to(0,-1)$.

Since in each case, there is only one possible limit for $(Per(A_{n'}))$ this proves that $(Per(A_n))$ converges to this limit. Proposition 8 is then established.

5.3 Conclusion

We use the preceding subsections to extend Per to $\widetilde{\mathcal{P}}$ then we get a continuous map on $\widetilde{\mathcal{P}}$. First we have the following remark.

Proposition 9. *The period map does not vanish on the boundary of* $\widetilde{\mathcal{P}}$

Proof. The only points where Per can vanish are points in the vertices and

if $Per(A) = 0$ we have $\overrightarrow{\gamma(\theta - \frac{\pi}{2})\gamma(\theta + \frac{\pi}{2})} = 0$ for some strictly convex curve γ. But γ on $(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$ is a graph over a straight-line then the above vector can not be 0. $\square$

This proves that we can compute the degree of the period map along the boundary of $\widetilde{\mathcal{P}}$.

Theorem 7. *The degree of the period map along the boundary of* $\widetilde{\mathcal{P}}$ *is* $-(r-1)$

Proof. The edges of $\mathcal{P}$ does not contribute toward the degree. So only the behaviour at the vertices is interesting for the degree. Let us compute the contribution of the vertex P_1; we use the notation of the preceding section. Since the curve γ is strictly convex, the map $\theta \mapsto \frac{\overrightarrow{\gamma(\theta - \frac{\pi}{2})\gamma(\theta + \frac{\pi}{2})}}{|\gamma(\theta - \frac{\pi}{2})\gamma(\theta + \frac{\pi}{2})|}$ is a monotone map. Let θ be in $[0, \alpha]$, the unit vector tangent to γ at $\gamma(\theta)$ is $(\cos\theta, \sin\theta)$ which turns in the clockwise sense, when θ increases. Besides, for $0 < \theta < \alpha$, $\gamma([\theta - \pi/2, \theta + \pi/2])$ is a graph over a straight-line generated by $(\cos\theta, \sin\theta)$. This implies that $Per(P_1, \theta) \cdot (\cos\theta, \sin\theta)$ never vanishes ((P_1, θ) is a point of the boundary of $\widetilde{\mathcal{P}}$); by looking at the behaviour for small θ we have $Per(P_1, \theta) \cdot (\cos\theta, \sin\theta) \geq 0$. Besides, for $\theta = 0$, the basis composed of $(1, 0)$ and $Per(P_1, 0)$ is an indirect one and, for $\theta = \alpha$, the basis $((\cos\alpha, \sin\alpha), Per(P_1, \alpha)$ is a direct one. Then, when θ increases from 0 to α, $Per(P_1, \theta)$ describes an angle α (since the unit tangent describes this angle) plus π (since the basis composed of the unit tangent at $\gamma(\theta)$ and $Per(P_1, \theta)$ is an orthonormal indirect one for $\theta = 0$ and an orthonormal direct one for $\theta = \alpha$). Since, when we describe $\partial\widetilde{\mathcal{P}}$ in the clockwise sense, θ decreases, the contribution of the vertex P_1 towards the degree is $-\frac{1}{2\pi}(\alpha + \pi)$.

The degree is then $-\dfrac{1}{2\pi}(r\pi + \displaystyle\sum_{i=0}^{r-1} \alpha_i)$ where α_i is the inner angle at the vertex P_i. Then by applying Gauss-Bonnet Theorem to $\mathcal{P}$, $\displaystyle\sum_{i=0}^{r-1} \alpha_i = r\pi - 2\pi$ and then the degree is $-(r-1)$. $\square$

Theorem 7 then proves that the degree of the period map is non zero and then there exists $A \in \overset{\circ}{\mathcal{P}}$ such that $Per(A) = 0$ and then Theorem 2 is proved.

6 An other example of solution the Plateau problem at infinity

Theorem 2 gives a wide class of solutions of the Plateau problem at infinity with genus 1. But it gives no example of polygon which is the the flux polygon of an r-noid of genus 1 but not the flux polygon of an r-noid of genus 0. Corollary 4 gives such polygons. In fact we study the case where the multi-domain with cone singularity bounded by the polygon is invariant under a "rotation".

Theorem 8. *Let V be a polygon that bounds a multi-domain with cone singularity (D, Q, ψ). We suppose that D satisfies the hypothesis H and that there exists an isometry h of D such that $\psi \circ h = R \circ \psi$ where R is the rotation in $\mathbb{R}^2$ with center $\psi(Q)$ and angle $\alpha \in (0, 2\pi)$. Then the period vector associated to D vanishes and there exists an Alexandrov-embedded r-noid with genus 1 and horizontal ends having V as flux polygon.*

Proof. From Construction 1 and 3, we have a multi-domain with logarithmic singularity $(\Omega, \mathcal{Q}, \varphi)$ and Theorem 3 gives us a function u on Ω which is unique up to an additive constant. By construction, the map h can be lifted to Ω to a map $\tilde{h}$ which is an isometry of Ω such that $\varphi_{\psi(Q)} \circ \tilde{h} = R \circ \varphi_{\psi(Q)}$. Since $\tilde{h}$ is an isometry of Ω, $u \circ \tilde{h}$ is a solution of the same Dirichlet problem as u then $u \circ \tilde{h} = u + c$ with $c \in \mathbb{R}$. Then the two equations $\varphi_{\psi(Q)} \circ \tilde{h} = R \circ \varphi_{\psi(Q)}$ and $u \circ \tilde{h} = u + c$ imply that:

$$\tilde{h}^*(\mathrm{d}X_1^*, \mathrm{d}X_2^*) = R(\mathrm{d}X_1^*, \mathrm{d}X_2^*)$$

with $\mathrm{d}X_1^*$ and $\mathrm{d}X_2^*$ the 1-forms associated to u as in Subsection 4.1. Then, if Γ is a lift of a generator of $\pi_1(D \backslash \{Q\})$, we have:

$$\begin{aligned}
\int_\Gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*) &= \int_{\tilde{h}(\Gamma)} (\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\
&= \int_\Gamma \tilde{h}^*(\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\
&= \int_\Gamma R(\mathrm{d}X_1^*, \mathrm{d}X_2^*) \\
&= R\left(\int_\Gamma (\mathrm{d}X_1^*, \mathrm{d}X_2^*)\right)
\end{aligned}$$

The first equality is due to the fact that Γ and $\tilde{h}(\Gamma)$ are lifts of two closed pathes that give the same generator of $\pi_1(D \backslash \{Q\})$. Besides R has a unique fixed point, since $\alpha \in (0, 2\pi)$, which is 0. Then the period vector vanishes.

$\square$

We then can give some examples of polygons V that satify this condition; we consider the case where V is a regular convex polygon or a regular star polygon (see [Cox]).

Corollary 4. *Let q and r be in $\mathbb{N}^*$ such that* $\gcd(q,r)=1$ *and* $2q<r$. *We note, for $i=1,\dots,r+1$, $P_i=e^{2(i-1)\sqrt{-1}\frac{q}{r}\pi}\in\mathbb{C}=\mathbb{R}^2$ ($P_1=P_{r+1}$) (see Figure 5). Then there exists an Alexandrov-embedded r-noid of genus* 1 *and horizontal ends with $(\overrightarrow{P_1P_2},\overrightarrow{P_2P_3},\dots,\overrightarrow{P_rP_{r+1}})$ as flux polygon.*

Proof. The idea of the proof is to build a multi-domain with cone singularity bounded by the above polygon and satisfying the hypothesis of Theorem 8. Let $\mathbb{R}^2$ be identified with $\mathbb{C}$. We note T the set of $(\rho,\theta)\in\mathbb{R}_+\times[0,\pi]$ such that $\rho e^{i\theta}$ is in the triangle P_1P_2O, where O is the origin. We note D the set of (ρ,θ) in $\mathbb{R}^+\times[0,2q\pi]$ such that (ρ,θ) is in D if (ρ,θ') is in T where $\theta=n\left(2\frac{q}{r}\pi\right)+\theta'$ is the only writing with $n\in\mathbb{Z}$ and $\theta'\in[0,2\frac{q}{r}\pi)$ (like an euclidean division of θ by $2\frac{q}{r}\pi$). We remark $\{0\}\times[0,2q\pi]\subset D$ and $(\rho,0)\in D$ iff $(\rho,2q\pi)\in D$ iff $\rho\in[0,1]$. Because of these remarks, we can consider the polar metric on D (we identify all the points $(0,\theta)$ and call the new point Q) and indentify the point $(\rho,0)$ with $(\rho,2q\pi)$ for $0\le\rho\le1$. Then if we consider on D the map $\psi:(\rho,\theta)\mapsto(\rho\cos\theta,\rho\sin\theta)$, the triplet (D,Q,ψ) is a multi-domain with cone singularity which is bounded by the polygon $(\overrightarrow{P_1P_2},\overrightarrow{P_2P_3},\dots,\overrightarrow{P_rP_{r+1}})$. The angle of D is $2q\pi$. Besides on D the map

$$h:(\rho,\theta)\longmapsto\begin{cases}(\rho,\theta+2\frac{q}{r}\pi) & \text{if } \theta\le 2q\pi-2\frac{q}{r}\pi\\ (\rho,\theta-2\frac{r-1}{r}q\pi) & \text{if } \theta\ge 2q\pi-2\frac{q}{r}\pi\end{cases}$$

is well defined and is an isometry of D. Besides $\psi\circ h=R\circ\psi$ with R the rotation of center O and angle $2\frac{q}{r}\pi<\pi$. Then we can apply Theorem 8. $\square$

When $q=1$, the polygon is a convex regular polygon and the multi-domain D that we build is in fact an immerssed polygonal disk, then Theorem 2 gives also the result but what we now know is that in the proof of Theorem 2 we can choose A as the isobarycenter of the polygon.

When $q>1$, the polygon $(\overrightarrow{P_1P_2},\overrightarrow{P_2P_3},\dots,\overrightarrow{P_rP_{r+1}})$ does not bound an immersed polygonal disk. Then we get new examples of polygons which are flux polygons of an Alexandrov-embedded r-noid with genus 1.

Remark 7. In [JM], L. P. Jorge and W. H. Meeks give Weierstrass data for r-noid with genus 0 and horizontal ends having as flux polygon the polygon studied in Corollary 4 with $q=1$. Then Corollary 4 gives r-noids similar

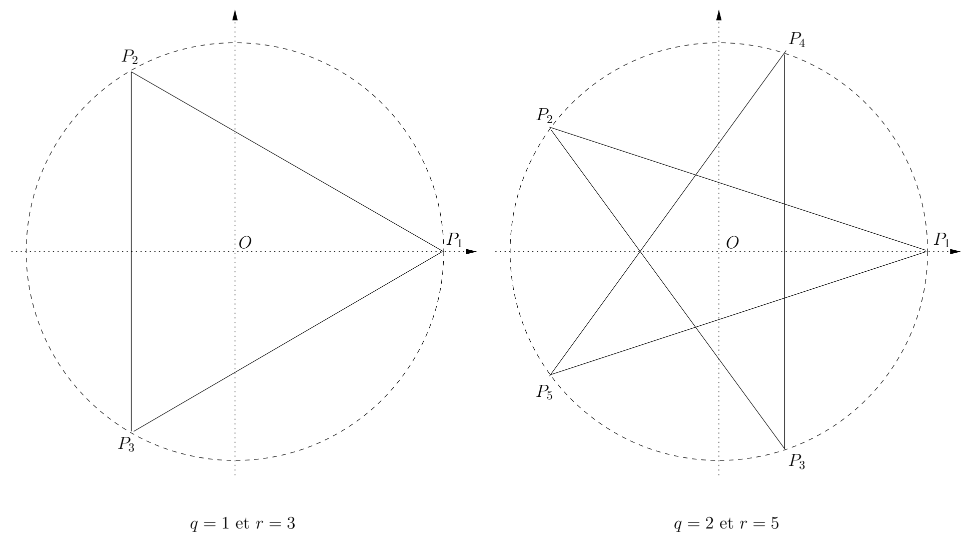

Figure 5: Examples of polygons studied in Corollary 4

to the Jorge-Meeks examples for the genus 1 case. In fact this examples are known by H. Karcher which gave in [Ka] Weierstrass data for some r-noids which correspond to the ones we have just built. These Weierstrass data are expressed in terms of the Weierstrass $\mathfrak{p}$-function.

A Convergence of sequences of solution of (MSE) and line of divergence

The aim of this appendix is to explain some results on the convergence of sequences of solutions of the minimal surface equation.

Let (u_n) be a sequence of solution of (MSE) on a multi-domain D (D has possibly a cone or logarithmic singularity), the first result is then

Proposition A.1. *If (u_n) is an uniformly bounded sequence on D, there exists a subsequence which converges to a solution of* (MSE) *on D. The convergence is uniform on every compact subset of D.*

This result proves that if the sequence $(||\nabla u_n||)$ is uniformly bounded on every compact subset of D there exists a subsequence $(u_{n'} - u_{n'}(P))$, where P is a point in D, which converges to a solution of (MSE) on D (D is connected). We then can define the domain of convergence of the sequence

(v_n) as the set of the points $P \in D$ where the sequence $(||\nabla u_n(P)||)$ is bounded, we note this set $\mathcal{B}(u_n)$. We know that $\mathcal{B}(v_n)$ is an open subset of D such that $(||\nabla u_n||)$ is uniformly bounded on every compact subset of $\mathcal{B}(u_n)$ (see [Ma1]). Then, on every connected component of the domain of convergence, we can make converge a subsequence $(u_{n'} - u_{n'}(P))$. In fact, in our paper we often write that a subsequence $(u_{n'})$ converges instead of $(u_{n'} - u_{n'}(P))$ but, since the value on the boundary are often infinite, this does not matter; when it is necessary we use $(u_{n'} - u_{n'}(P))$. The question is to understand the domain of convergence, in fact we shall understand $D\backslash\mathcal{B}(u_n)$.

Let P be in $D\backslash\mathcal{B}(u_n)$, then for a subsequence $W_{n'} \to +\infty$ and since the normal to the graph at the point over P is

$$N_{n'}(P) = \left(\frac{p_{n'}}{W_{n'}}(P), \frac{q_{n'}}{W_{n'}}(P), -\frac{1}{W_{n'}}(P)\right)$$

(we use euclidean coordinates in a neighborhood of P) we can suppose that $N_{n'}(P) \to N$ where N is a unit horizontal vector. We then have the following result.

Theorem A.1. *Let (D,ψ) be a multi-domain. Let (u_n) be a sequence of solutions of* (MSE) *on D. Let $P \in D$ and N be a unit horizontal vector and L the geodesic of D passing by P and normal to N. If the sequence $(N_n(P))$ converges to N, then $N_n(A)$ converges to N at every point A of L*

Since D is locally isometric to $\mathbb{R}^2$ we can see N as a vector in $\mathbb{R}^2$ and then the vector N is well defined at all the points of D in fact this definition coincides with the parallel transport of N. The proof of this theorem is in [Ma1].

Then in our situation $N_{n'}(P)$ is converging to a unit horizontal vector then $N_{n'}$ converges to this vector along a straight-line L. Such a line L is called a *line of divergence* of the sequence since L must be included in $D\backslash\mathcal{B}(u_n)$. Then the set $D\backslash\mathcal{B}(u_n)$ is an union of geodesics of D and the question is what are the possible lines of divergence: the following lemmas give some tools to answer to this discussion.

First, we observe that the existence of a line of divergence has a consequence on the sequence of 1-forms $\mathrm{d}\Psi_{u_n}$. If $N_n \to N$ along a line of divergence and T is a segment included in this line of divergence then $\int_T \mathrm{d}\Psi_{u_n}$ converges to $|T|$ the length of T if the orientation of T is such that N is the right-hand normal. Since (Ψ_{u_n}) is a sequence of 1-Lipschitz continuous function, we can always take a subsequence and suppose that it has a limit

Ψ, then the above remark on $\mathrm{d}\Psi_{u_n}$ allows us to make some calculations on Ψ.

As in Figure 3, when we make a picture to explain the convergence of a sequence of solutions of (MSE), we draw the limit normal along the lines of divergence to explain the asymptotical behaviour.

We then have the two following results concerning the lines of divergence and the convergence of sequences of solutions of (MSE)

Lemma A.1. *Let (u_n) be a sequence of solutions of* (MSE) *on $[0,1]^2$ such that, for all n, the function u_n tends to $+\infty$ on $\{1\}\times]0,1[$. Then, no line of divergence of the sequence (u_n) has $(1,\frac{1}{2})$ as end-point.*

Proof. Let us suppose that there exists such a line of divergence L. We note $A=(1,0)$, $B=(1,\frac{1}{2})$ and $C=(1,1)$. Let us consider a point $M\in L$. We suppose that the limit normal along L is such that the basis composed of $\overrightarrow{MB}$ and the limit normal is direct. Since $\mathrm{d}\Psi_{u_n}$ is closed, we then have:

$$\int_{[A,B]}\mathrm{d}\Psi_{u_n}+\int_{[B,M]}\mathrm{d}\Psi_{u_n}+\int_{[M,A]}\mathrm{d}\Psi_{u_n}=0$$

Since u_n takes the value $+\infty$ on $[A,B]$, this equality proves that:

$$|AB|+\int_{[B,M]}\mathrm{d}\Psi_{u_n}\leq|MA|$$

But, by our choice of limit normal, we have $\int_{[B,M]}\mathrm{d}\Psi_{u_n}\to|BM|$ for a subsequence, then $|AB|+|BM|\leq|MA|$ which contradicts the triangle inequality. If the limit normal is the opposite of the one we consider, we can do the same arguement with the triangle BCM. $\square$

Lemma A.2. *Let (u_n) be a sequence of solutions of* (MSE) *on $[0,1]^2$ such that (u_n) converges in the interior of the square to a solution u and such that we are in one of the following two cases*

- *for all n, the function u_n tends to $+\infty$ on $\{1\}\times]0,1[$ or,*
- *for all n, u_n is the restriction to $[0,1]^2$ of a solution v_n of* (MSE) *defined on $[0,2]\times[0,1]$ and for $y\in]0,1[$ we have $N_n(1,y)\to(1,0,0)$.*

Then, the limit function u tends to $+\infty$ on $\{1\}\times]0,1[$.

Proof. let $0<\varepsilon<\frac{1}{2}$, for $x\in[0,1]$ we note $A_x=(x,\varepsilon)$ and $B_x=(x,1-\varepsilon)$. Since $\mathrm{d}\Psi_{u_n}$ is closed, for $x<1$ we have:

$$\int_{[A_x,B_x]}\mathrm{d}\Psi_{u_n}-\int_{[A_1,B_1]}\mathrm{d}\Psi_{u_n}=\int_{[B_x,B_1]}\mathrm{d}\Psi_{u_n}+\int_{[A_1,A_x]}\mathrm{d}\Psi_{u_n}$$

Then, by passing to the limit and using that $\int_{[A_1,B_1]} \mathrm{d}\Psi_{u_n} \to 1-2\varepsilon$ in the two cases, we obtain that:

$$\left| \int_{[A_x,B_x]} \mathrm{d}\Psi_u - (1-2\varepsilon) \right| \leq 2(1-x)$$

This proves that $\mathrm{d}\Psi_u = \mathrm{d}y$ on $[A_1, B_1]$. Let $x \in]0,1[$ and v be the solution of (MSE) on $A_xB_xB_1A_1$ such that v tends to $+\infty$ on $[A_1, B_1]$ and $v = u$ on the rest of the boundary we shall prove that $u = v$.

If $u \neq v$ there exist $\eta \neq 0$ such that $\Omega = \{u - v > \eta\}$ is non-empty. The boundary of Ω is composed of one part included in the interior of $A_xB_xB_1A_1$ and one part included in $[A_1, B_1]$. Since $\mathrm{d}\Psi_u$ and $\mathrm{d}\Psi_v$ are closed we have

$$\int_{\partial\Omega} \mathrm{d}\Psi_u - \mathrm{d}\Psi_v = 0$$

But on the part of the boundary included in $[A_1, B_1]$ the integral is 0 since $\mathrm{d}\Psi_u = \mathrm{d}y = \mathrm{d}\Psi_v$ and on the part included in the interior of $A_xB_xB_1A_1$ the integral is negative by Lemma 2 in [CK]. Then we have a contradiction and $u = v$ then u goes to $+\infty$ on $[A_1, B_1]$. □

B Convex curves in $\mathbb{R}^2$

A convex curve in $\mathbb{R}^2$ is a curve such that its geodesical curvature has always the same sign. A curve is strictly convex if this geodesical curvature is positive or negative.

If a curve $c : s \mapsto c(s) \in \mathbb{R}^2$ is convex, the map with value in $\mathbb{S}^2$ that associates to the parameter s the normal to c at $c(s)$ is a monotone map and, if c is strictly convex, the above map is strictly monotone. This implies that, if $c : I \to \mathbb{R}^2$ is a strictly convex curve, there exists $h : J \to I$ a diffeomorphism such that for $\beta \in J$ the normal to c at the point $c \circ h(\beta)$ is $(\sin\beta, -\cos\beta)$. We then say that c is parametrized by its normal.

Proposition B.1. *Let $c : I \to \mathbb{R}^2$ be a strictly convex curve parametrized by its normal. If $I \subset (\beta_0 + \frac{\pi}{2}, \beta_0 + \frac{3\pi}{2})$, the curve c is a graph over the straight line $y\cos\beta_0 - x\sin\beta_0 = 0$.*

Proof. We suppose that $\beta_0 = 0$. If $c(\beta) = (x(\beta), y(\beta))$, the map $\beta \mapsto x(\beta)$ is a local diffeomorphism since the second coordinate of the normal is always positive. Besides $\beta \mapsto x(\beta)$ is injective: if $x(\beta') = x(\beta'')$ there would exist, by Rolle's Theorem, $\beta \in (\beta', \beta'')$ such that the normal at $c(\beta)$ is horizontal, it is impossible. This proof that $\beta \mapsto x(\beta)$ is a global diffeomorphism then c is a graph over $y = 0$. □

Proposition B.2. *Let c be a strictly convex curve parametrized by its normal on $(\theta - \varepsilon, \theta + \beta]$ with $\beta \leq \frac{\pi}{2}$. Let v be the unit tangent vector to c at $c(\theta)$ and n' the unit vector which is the normal to c at $c(\theta)$ if the curvature is positive or the opposite of the normal if the curvature is negative. Let w be the unit vector $\cos\beta v + \sin\beta n'$. Then the curve $c([\theta, \theta + \beta])$ is in the angular sector delimited by the two half straight-lines with $c(\theta)$ as end-point and respectively generated by v and w (see Figure 6).*

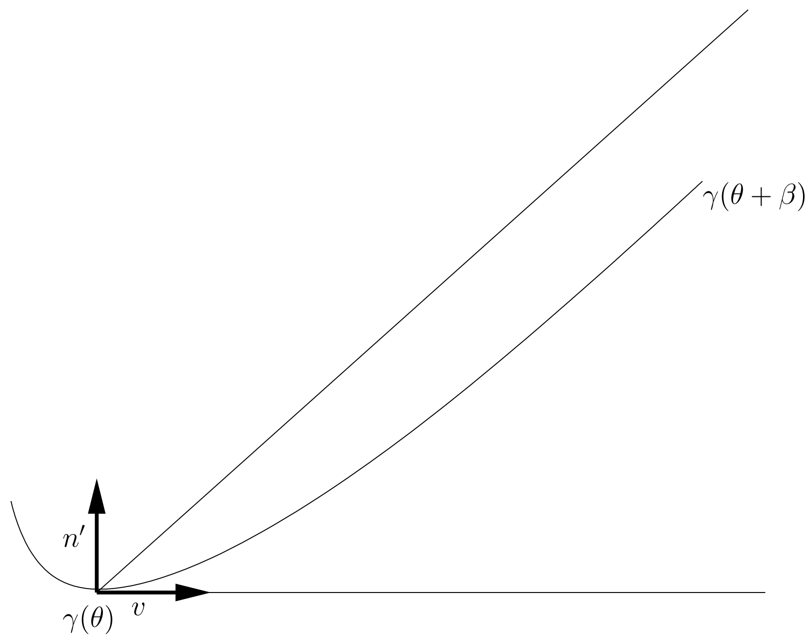

Figure 6: the situation of Proposition B.2

Proof. We can suppose that $\theta = 0$, the curvature is negative and $c(0) = (0, 0)$. In this case $c([0, \beta])$ is the graph of a function f over a segment $[0, a]$ by Proposition B.1. By hypotheses, f is a convex function and $f'(0) = 0$. This implies first that $f \geq 0$. If $\beta = \frac{\pi}{2}$ the proposition is proved. If $\beta < \frac{\pi}{2}$ and $c([0, \beta])$ is not in $\{(x, y)|\ x \geq 0,\ y \geq 0,\ y \leq x \tan\beta\}$ there is a parameter $b \in [0, a]$ such that $f(b) = b \tan\beta$ and then there exists $d < b$ such that $f'(d) = \tan\beta$. Since the normal map is injective on $[0, \beta]$, we have $(d, f(d)) = c(\beta)$ which is impossible since $d < a$. The proposition is then proved $\square$

Lemma B.1. *Let c be a strictly convex curve parametrized by its normal on $[\beta - \varepsilon, \beta)$. We suppose that the curve c is included in a compact of $\mathbb{R}^2$. Then $c(t)$ converges when $t \to \beta$.*

Proof. Since we are in a compact part of $\mathbb{R}^2$, it is enough to prove that there is only one cluster point for $c(t)$. Let us suppose that $c(t_n) \to p_1$ and $c(s_n) \to p_2$ and, for every n, $t_n < s_n < t_{n+1}$. We then apply Proposition B.2 on $[t_n, \beta)$ and $[s_n, \beta)$. This proves that p_1 is in an angular sector with vertex $c(s_n)$ and $\beta - s_n$ as angle and p_2 is in an angular sector with vertex $c(t_n)$ and $\beta - t_n$ as angle. Letting n goes to $+\infty$, we obtain that p_1 is in the half straight-line with p_2 as end-point and generated by v (where v is the limit unit tangent vector) and p_2 is in the half straight-line with p_1 as end-point and generated by v. This situation is possible only if $p_1 = p_2$. □

Proposition B.3. *Let (c_n) be a sequence of stricly convex curves parametrized by their normal on $(\theta - \varepsilon, \theta + \varepsilon)$. We suppose that (c_n) converges to $\widetilde{c}$ on $(\theta - \varepsilon, \theta) \cup (\theta, \theta + \varepsilon)$ in the C^1 topology. Then:*

- *we have $\widetilde{c}(t) \to p_1$ when $t \to \theta$, $t < \theta$ and $\widetilde{c}(t) \to p_2$ when $t \to \theta$, $t > \theta$,*
- *as sets, $c_n(\theta - \varepsilon, \theta + \varepsilon)$ converges to $\widetilde{c}(\theta - \varepsilon, \theta) \cup \widetilde{c}(\theta, \theta + \varepsilon) \cup [p_1, p_2]$.*

If $p_1 = p_2$, we have moreover that (c_n) converges to $\widetilde{c}$ (that we extend by $\widetilde{c}(\theta) = p_1$) on $(\theta - \varepsilon, \theta + \varepsilon)$ in the C^1 topology.

Proof. ε is supposed to be small and we choose $\varepsilon' < \varepsilon$. We then apply Proposition B.2 to $c_n(\theta - \varepsilon', \theta + \varepsilon')$ at the points $c_n(\theta - \varepsilon')$ and $c_n(\theta + \varepsilon')$ and we get that, for every n, $c_n(\theta - \varepsilon', \theta + \varepsilon')$ is included in an angular sector of vertex $c_n(\theta - \varepsilon')$ and angle $2\varepsilon'$ and an angular sector of vertex $c_n(\theta + \varepsilon')$ and angle $2\varepsilon'$ (here, we apply Proposition B.2 to the curve c that we cover in the opposite sense). Letting n goes to $+\infty$, we get that $\widetilde{c}((\theta - \varepsilon', \theta) \cup (\theta, \theta + \varepsilon'))$ is included in the intersection of two angular sectors of angle $2\varepsilon'$, one has $\widetilde{c}(\theta - \varepsilon')$ as vertex the other has $\widetilde{c}(\theta + \varepsilon')$ as vertex. The intersection of this two angular sector is a compact; then, by Lemma B.1, p_1 and p_2 exists. We then have also proved that the cluster points of the sequence of curves (c_n) are $\widetilde{c}((\theta - \varepsilon, \theta) \cup (\theta, \theta + \varepsilon))$ and points included in the intersection of the two angular sectors. If we let ε' goes to 0 the intersections of the two sectors converge to the segment $[p_1, p_2]$ which have $(\sin\theta, -\cos\theta)$ as normal. We then must show that all the points of the segment $[p_1, p_2]$ are the limit of a sequence $(c_n(t_n))$.

We suppose now that $\theta = 0$. By Proposition B.1, the curves c_n are graphs over $\{y = 0\}$. Let a and b be the respective first coordinates of $\widetilde{c}(-\varepsilon/2)$ and

$\widetilde{c}(\varepsilon/2)$; we suppose $a < b$. Since $c_n \to \widetilde{c}$, we then can ensure that, over the segment $[a, b]$, the curves c_n are graphs for big n. We have $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$; since the segment $[p_1, p_2]$ is horizontal, $y_1 = y_2$. Besides by convergence of (c_n), $a < x_1 \leq x_2 < b$. Let $x \in [x_1, x_2]$; since c_n is a graph over $[a, b]$ for big n, there exist one parameter t_n such that $c_n(t_n)$ has x as first coordinate. Then the only possible cluster point for the sequence $(c_n(t_n))$ is (x, y_1) since every cluster point must have x as first coordinate and be in the segment $[p_1, p_2]$ or in the curve $\widetilde{c}$ but all this set is a graph over $\{y = 0\}$. This then proves that all the points of the segment $[p_1, p_2]$ is in the limit set.

If $p_1 = p_2$, (c_n) converges in the C^1 topology since the normal at the point $\widetilde{c}(\theta) = p_1$ is $(\sin\theta, -\cos\theta)$ by continuity. □

References

[ABR] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Graduate Texts in Mathematics **137** (Springer-Verlag, 2001).

[CK] P. Collin and R. Krust, Le problème de Dirichlet pour l'équation des surfaces minimales sur des domaines non bornés, Bul. Soc. Math. France. **119** (1991), 443–462.

[CR] C. Cosín and A. Ros, A Plateau problem at infinity for properly immersed sufaces with finite total curvature, Indiana Univ. Math. J. **50** (2001), 847–879.

[Cou] R. Courant, *Dirichlet's Principle, Conformal Mapping and Minimal Surface*, (Springer-Verlag, 1977).

[Cox] H. S. M. Coxeter, *Introduction to Geometry*, Wiley Classics Library (John Wiley & Sons Inc., 1989).

[Fu] W. Fulton, *Algebraic Topology, A First Course*, Graduate Texts in Mathematics **153** (Springer-Verlag, 1995).

[Hi] S. Hildebrandt, Boundary value problems for minimal surfaces, *Geometry V*, Encylopaedia Math. Sci. **90** (1997), 153–237.

[HK] D. Hoffman and H. Karcher, Complete embedded minimal surfaces of finite total curvature, *Geometry V*, Encylopaedia Math. Sci. **90** (1997), 5–93.

[JM] L. P. Jorge and W. H. Meeks III, The topology of complete minimal surfaces of finite totale Gaussian curvature, Topology **22** (1983), 203–221.

[JS] H. Jenkins and J. Serrin, Variational problems of minimal surface type II, Arch. Rational Mech. Anal. **21** (1966), 321–342.

[Ka] H. Karcher, Embedded minimal surfaces derived from Scherk's examples, Manuscripta Math. **62** (1988), 83–114.

[Ma1] L. Mazet, The Dirichlet problem for minimal surfaces equation and Plateau problem at infinity, to appear in J. Inst. Math. Jussieu.

[Ma2] L. Mazet, Quelques résultats de non-existence pour l'équation des surfaces minimales, preprint.

[Ni] J.C.C. Nitsche, On new results on the theory of minimal surfaces, Bull. Amer. Math. Soc. **71** (1965), 195–270.

[Os] R. Osserman, *A Survey On Minimal Surfaces*, Van Nostrand Math. Studies. (1969).

[PR] J. Pérez and A. Ros, Properly embedded minimal surfaces with finite total curvature, Lecture Notes in Math. **1775** (2002), 15–66.

[RSE] H. Rosenberg and R. Sa-Earp, The Dirichlet problem for the minimal surface equation on unbounded planar domains, J. Math. Pures Appl. **68** (1989), 163–183.

Laurent Mazet
Laboratoire Emile Picard (UMR 5580), Université Paul Sabatier,
118, Route de Narbonne, 31062 Toulouse, France.
E-mail: mazet@picard.ups-tlse.fr